Modeling Autocallable Structured Products

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Abstract

Since first introduced in 2003, the number of autocallable structured products in the U.S. has increased exponentially. The autocall feature immediately converts the product if the reference asset's value rises above a pre-specified call price. Because an autocallable structured product matures immediately if it is called, the autocall feature reduces the product's duration and expected maturity.

In this paper, we present a flexible Partial Differential Equation (PDE) framework to model autocallable structured products. Our framework allows for products with either discrete or continuous autocall dates. We value the autocallable structured products with discrete autocall dates using the finite difference method, and the products with continuous autocall dates using a closed-form solution. In addition, we estimate the probabilities of an autocallable structured-product being called on each call date. We demonstrate our models by valuing a popular autocallable product and quantify the cost to the investor of adding this feature to a structured product.

1 Introduction

Autocallable structured products (Fries and Joshi, 2008; Georgieva, 2005) have become increasingly common in recent years. Figure 1(a) and Figure 1(b) plot the number and aggregate face value of autocallable structured products issued between 2003 and 2010.[1] As the figures indicate, the number of issues increased sharply in 2007 and has continued to grow through 2010 at a 40% annual growth rate. In just the first six months of 2010

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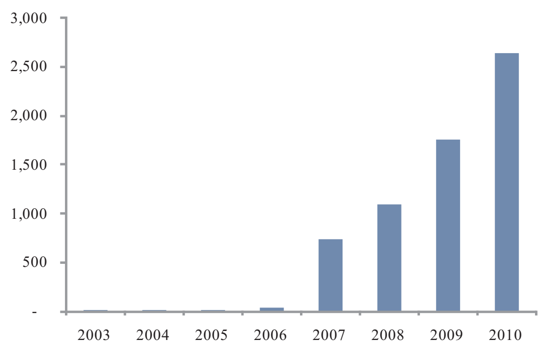
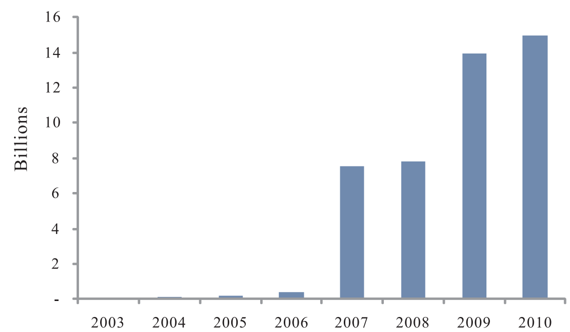
[1]The first autocallable structured product on record in the U.S. was issued by BNP Paribas on August 15, 2003.

there have been more than 2,500 autocallable products issued. The aggregate face value of newly issued autocallable structured products follows the same pattern, with a surge in 2007 and continued growth since then.

Figure 1: Number and Total Issue Size of Autocallable Structured Products, January 2003 - June 2010.



(a) Number of Autocallables Issued



(b) Aggregate Autocallable Issue Size

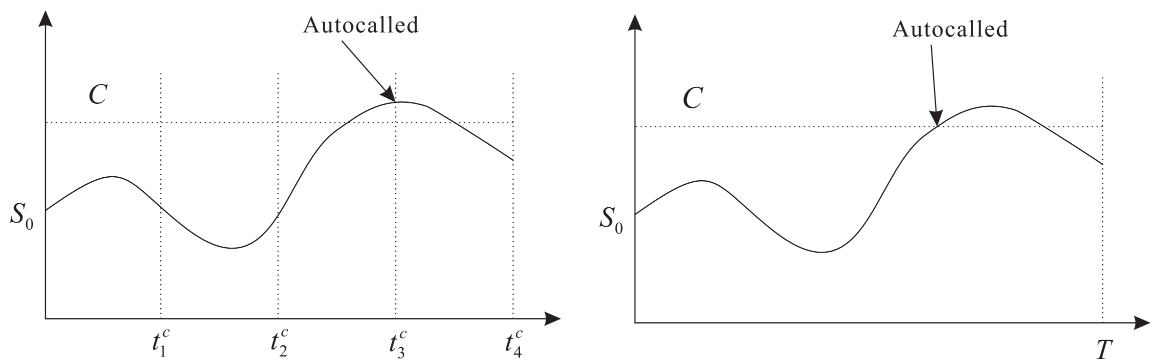
One reason for the rapid expansion of autocallable structured products is the ease with which the autocall feature can be attached to existing types of structured products (Baule et al., 2008; Bergstresser, 2008; Henderson and Pearson, 2010; Deng et al., 2010a). The autocall feature immediately converts the structured product if the reference asset's price reaches or exceeds a predefined level (the *call price*) on a call date.

In this paper we describe the autocall feature, explain how to value it, and show an example of the valuation methodology. We use this example to discuss the cost this feature can add to a structure product. We value autocallable structured products using a general Partial Differential Equation (PDE) approach. We set up the PDE using the Black-Scholes equation and add boundary conditions representing the product's features, including the autocall feature (Black and Schole, 1973; Wilmott et al., 1994).

We divide the autocallable structured products into two categories: products that have discrete call dates ("discrete autocallables") and products that have continuous call dates ("continuous autocallables"). Figure 2(a) and Figure 2(b) demonstrate graphically the difference between discrete and continuous autocallables. Both figures plot the same underlying stock price over time. The continuous autocallable is called immediately upon crossing the call price C, while the discrete autocallable must wait until t_3^c before it is called.[2] If the underlying stock price had dropped below C on t_3^c, the discrete autocallable would not have been called. Thus, holding all else equal, a continuous autocallable structured product is more likely to be called than a discrete one.

[2]Although we only consider constant call price in the paper, the methodologies are expandable to exponentially increasing call prices. Closed-form solutions are also available. The extension is analogous to valuing a barrier option with an exponentially varying barrier (see Kunitomo and Ikeda (1992) and Li (1998)).

Figure 2: An Autocall Event



(a) Discrete Callable

(b) Continuous Callable

An autocallable structured product is fundamentally similar to a reverse-convertible[3] that pays a high coupon payment while exposing the investor to the downside risk of the reference security. Although autocallable structured products tend to be issued for longer durations than reverse-convertibles, autocallable structured products can have shorter effective durations due to the embedded call feature.[4]

For example, a common autocallable structured product[5] would have the following payoffs: If the reference asset's price is above the call price on one of the call dates, it is immediately called, and pays a pre-specified fixed-rate return. If the reference asset's price is below the call price on every call date, the product is never called. In such a case, the investors receive the product's face value at maturity, if the final price of the reference asset is a above a predetermined threshold. If the final price is below the threshold, investors receive the same negative percentage return as the reference asset.

The paper proceeds as follows: In Section 2, we explain our valuation framework. Section 2.2 discusses autocallable structured products with discrete call dates, and Section 2.3 presents autocallable structured products with continuous call dates. Section 3 implements our valuation framework for a example autocallable structured product. We conclude in Section 4. In the appendix we explain the main features of popular autocallable structured-products.

[3]See for example Hernández et al. (2007) for an explanation of reverse-convertibles.

[4]See Arzac (1997), Chemmanur et al. (2006), and Chemmanur and Simonyan (2010) for a discussion of why investment banks issue mandatory convertibles and why investors purchase them.

[5]See the Appendix for a description of the main brands of autocallable structured products and their standard features.

2 Autocallable Structured Product Valuation Models

There are three main characteristics of the autocall feature that will affect the value of the structured product: the timing of the call dates, the probability of being called on each call date, and the determination of the payoff at maturity. In this section we set up the valuation of autocallable structured products as a Partial Differential Equation (PDE) problem. The PDE problem is general enough to be used on both discrete and continuous autocalls.

2.1 Modeling Autocallable Structured Products Using PDE

Our valuation model follows the Black-Scholes framework with risk-neutral assumptions. The reference asset's price is a generalized Brownian motion

$$dS_t = (r - q)S_t dt + \sigma S_t dW_t, \tag{1}$$

where r is the risk-free rate, q is the dividend yield, and σ is the volatility of the price process.[6] If we assume the price of the structured product $V(S,t)$ is a function of time $t \in [0,T]$ and the reference asset's price $S \in [0,\infty)$. The Black-Scholes formula implies that a structured product's dynamic value can be expressed as the following PDE:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - (r + C\bar{D}S)V = 0, \tag{2}$$

where $C\bar{D}S$ is the credit default swap (CDS) spread of the issuer.[7]

Many different structured product features can be modeled as variations on the PDE (Equation (2)). For example, when the structured product is not called, the payoff at maturity $f(S_T)$ is typically a function of the value of the reference asset at maturity:[8]

$$V(S_T, T) = f(S_T).$$

Embedded call and put options and the autocall feature can all be modeled as boundary conditions (Deng et al., 2010a). The autocall feature's boundary condition is

$$V(C,t) = P_t, \text{ for } t \in \mathcal{T}_{\mathcal{C}}, \tag{3}$$

where C is the time-independent call price, P_t is the final payoff if the note is called, and $\mathcal{T}_{\mathcal{C}}$ is a set of discrete or continuous call dates. Note that once the autocall is triggered, the

[6]Throughout this paper, we assume r, q, and σ are constant and continuously compounded over the product's term $[0,T]$. For simplicity, we omit the subscript t from S_t.

[7]Structured products are unsecured debt securities, and hence lose value if the issuer defaults. It is therefore essential to include the issuer's credit risk $C\bar{D}S$ in the PDE to calculate the structured product's present value (Hull, 2008; Deng et al., 2010a).

[8]For simplicity and without loss of generality, we assume the initial principal of a structured product is equal to the reference asset's initial value S_0.

structured product matures immediately and the final payout is P_t. Autocalled structured products typically pay out a fixed rate of return. Therefore, the payoff follows

$$P_t = He^{Bt},$$

where B is the rate of return, and H is a constant.

2.2 Valuing Autocallable Structured Products with Discrete Call Dates

Most discrete autocallables do not have a closed-form solution. Instead, the PDE is solved and the products are valued via numerical methods such as the finite-difference method (Zvan et al., 2000).

For discrete autocallables, the boundary conditions of the PDE are the equations

$$V(C,t) = P_t \text{ for all } t \in \mathcal{T}_\mathcal{C},$$

$$V(0,t) = f(0)e^{-(r+C\bar{D}S)(T-t)}.$$

The first condition requires that the product's value never exceeds the autocall payout on a call date. The second condition, is the boundary condition, and guarantees that if the reference asset's price hits 0 it will remain 0. For tractability, we define it using a general function $f(0) = 0$. This boundary condition is necessary as it guarantees that the structured product cannot ever be called in case the reference asset becomes worthless.

The first step in solving the PDE is to simplify the complex notation and transform the equation into a standard heat equation. Using a 'dimensionless' change of variables similar to Wilmott et al. (1994) and Hui (1996), we transform the variables $\{S, t, V(S,t)\}$ into $\{x, \tau, u(x,\tau)\}$ as follows

$$S = Ce^x, \quad t = T - \frac{2\tau}{\sigma^2}, \quad V(S,t) = Ce^{\alpha x + \beta\tau}u(x,\tau) + f(0)\, e^{-(r+C\bar{D}S)(T-t)},$$

where the constants are

$$k_1 = \frac{2(r-q)}{\sigma^2}, \quad \alpha = -\frac{1}{2}(k_1 - 1), \quad \beta = -\alpha^2 - \frac{2(r + C\bar{D}S)}{\sigma^2}.$$

After the change of variables, the Black-Scholes equation is reduced to a heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad \text{for } -\infty < x < 0, \quad \tau > 0, \tag{4}$$

the boundary conditions become

$$u(-\infty,\tau) = 0, \quad u\,(0,\tau) = C^{-1}e^{-\beta\tau}(P_t - f(0)e^{-(r+C\bar{D}S)\frac{2\tau}{\sigma^2}}) \quad \text{for } T - \frac{2\tau}{\sigma^2} \in \mathcal{T}_\mathcal{C}$$

and the initial condition becomes[9]

$$u(x,0) = C^{-1}e^{-\alpha x}\left(f(Ce^{x}) - f(0)\right), \text{ for } -\infty < x < 0.$$

To further simplify notation we denote

$$h_1(\tau) = C^{-1}e^{-\beta\tau}(P_t - f(0)e^{-(r-C\bar{D}S)\frac{2\tau}{\sigma^2}}) \quad \text{and} \quad h_2(x) = C^{-1}e^{-\alpha x}\left(f(Ce^{x}) - f(0)\right),$$

reducing the boundary conditions and initial condition to

$$u(-\infty,\tau) = 0, \quad u(0,\tau) = h_1(\tau) \quad \text{for } T - \frac{2\tau}{\sigma^2} \in \mathcal{T}_C, \quad u(x,0) = h_2(x). \tag{5}$$

The finite difference method allows to discretize the domain of the function $u(x,\tau)$, which is a plane $(x,\tau) \in (-\infty, 0] \times [0, \frac{T\sigma^2}{2}]$. We discretize the plane into an $N \times M$ grid, where the size of each grid block is $\delta x \times \delta\tau$. Because x has no lower bound, we can assign x an arbitrarily large minimum value of $-N\,\delta x$. The bounds on τ require that $\delta\tau$ satisfy

$$M\,\delta\tau = \frac{T\sigma^2}{2}.$$

Generally speaking, the accuracy of the valuation increases as δx and $\delta\tau$ get smaller. $\delta\tau$ is typically set to correspond to one trading day, such that $\delta t = \frac{2\delta\tau}{\sigma^2} = 1/250$ of a year.

There are three finite difference methods: the explicit finite difference method, the implicit finite difference method, and the Crank-Nicolson method. The methods differ in how they approximate the derivatives $\frac{\partial u}{\tau}$ and $\frac{\partial^2 u}{x^2}$. In this example, we use the explicit finite difference method, which approximates the derivatives as

$$\frac{\partial u}{\tau} \sim \frac{u_n^{m+1} - u_n^m}{\delta\tau},$$

$$\frac{\partial^2 u}{x^2} \sim \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2}.$$

Solving Equation (4) with the conditions in Equation (5) is equivalent to solving

$$\frac{u_n^{m+1} - u_n^m}{\delta\tau} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\delta x)^2}, \quad 0 < n < N, \quad 0 < m < M$$

with the conditions

$$u_n^0 = h_2(-n\,\delta x), \quad 0 \le n \le N,$$
$$u_0^m = 0, \quad 0 < m < M,$$
$$u_N^m = h_1(m\,\delta\tau), \quad \text{for } T - 2(m\,\delta\tau)/\sigma^2 \in \mathcal{T}_C.$$

[9]The change of variables converts the final condition into an initial condition.

The formula updating $m\delta\tau$ to $(m+1)\delta\tau$ is therefore

$$u_n^{m+1} = u_n^m + \frac{\delta\tau}{(\delta x)^2}\left(u_{n+1}^m - 2u_n^m + u_{n-1}^m\right), \quad 0 < n < N, \quad 0 < m < M.$$

The solution is derived iteratively from $m = 0 \to M$, which corresponds to $t = T \to 0$. For the convergence and stability of the explicit finite difference method, we require that $\frac{\delta\tau}{(\delta x)^2} \leq \frac{1}{2}$. Once all of the u_n^M, for $n = 1, 2, \ldots, N$ are derived, we can approximate $u\left(x, \frac{T\sigma^2}{2}\right)$ for every x. By reversing the change of variables, we can use $u\left(x, \frac{T\sigma^2}{2}\right)$ to finally solve the original function $V(S,t)$ at $t = 0$.

An alternative probability approach to valuing discrete autocallables

Another way to estimate the value of a discrete autocallable is by calculating the probability of the autocall being exercised on each call date, and then use the probability at each date to value the structured product. Let $p_i, i = 1, \ldots, n$ be the probability of the autocall being exercised at time t_i^c. The probability of the autocall never being exercised is then $1 - \sum_{i=1}^n p_i$, where each p_i is conditional on the autocall not being exercised at any previous call date $(t_1^c, \ldots, t_{i-1}^c)$. Recall that the conditional distribution of $S_{t_i^c}|S_{t_{i-1}^c}$ follows a lognormal distribution

$$S_{t_i^c} = S_{t_{i-1}^c} e^{(r-q-\frac{1}{2}\sigma^2)\Delta t_i^c + \sigma\Delta\sqrt{t_i^c}W_i},$$

where Δt_i^c is the time between call dates $\Delta t_i^c = t_i^c - t_{i-1}^c$ and $W_i, i = 1, \ldots, n$ are i.i.d. standard normal variables. To simplify notation, we use $X_i = (r-q-\frac{1}{2}\sigma^2)\Delta t_i^c + \sigma\Delta\sqrt{t_i^c}W_i$ to represent the continuously compounded return from t_{i-1}^c to t_i^c. This means the ending stock price S_T can be written as

$$\begin{aligned} S_T &= S_0 e^{\sum_{i=1}^n (r-q-\frac{1}{2}\sigma^2)\Delta t_i^c + \sigma\Delta\sqrt{t_i^c}W_i} \\ &= S_0 e^{\sum_{i=1}^n X_i}. \end{aligned}$$

Because of the price's Markov property, the X_i's are pairwise independent. Furthermore, if Δt_i^c is a constant, the X_i's are i.i.d. normal variables. The probability of the autocall being exercised at time t_i^c can now be written as

$$\begin{aligned} p_i &= Prob\left(S_{t_j^c} < C, j = 1, 2, \ldots, i-1, \text{ and } S_{t_i^c} \geq C\right) \\ &= Prob\left(\sum_{k=1}^j X_k < \log\left(\frac{C}{S_0}\right), j = 1, 2, \ldots, i-1, \text{ and } \sum_{k=1}^i X_k \geq \log\left(\frac{C}{S_0}\right)\right) \\ &= \underset{\substack{\sum_{k=1}^j x_k < \log\left(\frac{C}{S_0}\right), j=1,2,\ldots,i-1, \\ \sum_{k=1}^i x_k \geq \log\left(\frac{C}{S_0}\right)}}{\int \cdots \int} g(x_1, \ldots, x_n) dx_1 dx_2 \cdots dx_n, \end{aligned}$$

where $g(x_1, \ldots, x_n)$ is the joint probability density function (PDF) of $X_1, \ldots, X_n$. Because the X_i's are independent, the joint PDF can be expressed as the product of each X_i's individual PDF.

We can now estimate the product's present value as the discounted expected cash flows, where the cash flow probabilities are the p_i we just calculated.

$$\begin{aligned}
V(S_0, 0) &= \sum_{i=1}^{n} e^{-(r+C\bar{D}S)t_i^c} p_i P_{t_i^c} + \\
&\quad e^{-(r+C\bar{D}S)T} \int \cdots \int_{\sum_{k=1}^{j} x_k < \log\left(\frac{C}{S_0}\right), j=1,2,\ldots,n} f(S_T) g(x_1, \ldots, x_n) dx_1 \cdots dx_n \\
&= \sum_{i=1}^{n} e^{-(r+C\bar{D}S)t_i^c} p_i P_{t_i^c} + \\
&\quad e^{-(r+C\bar{D}S)T} \int \cdots \int_{\sum_{k=1}^{j} x_k < \log\left(\frac{C}{S_0}\right), j=1,2,\ldots,n} f(S_0 e^{\sum_{i=1}^{n} x_i}) g(x_1, \ldots, x_n) dx_1 \cdots dx_n
\end{aligned} \tag{6}$$

If the structured product's payoff at maturity is constant $f(S_T) = P_T$, the equation can be further reduced to

$$V(S_0, 0) = \sum_{i=1}^{n} e^{-(r+C\bar{D}S)t_i^c} p_i P_{t_i^c} + e^{-(r+C\bar{D}S)T} \left(1 - \sum_{i=1}^{n} p_i\right) P_T. \tag{7}$$

2.3 Valuing Autocallable Structured Products with Continuous Call Dates

For a continuous autocallable structured product, the boundary conditions of the PDE are continuous equations

$$V(C, t) = P_t, \quad V(0, t) = f(0) e^{-(r+C\bar{D}S)(T-t)}.$$

We apply the change of variables and simplifications from Section 2.2, yielding the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \qquad \text{for } -\infty < x < 0, \quad \tau > 0, \tag{8}$$

with the boundary conditions

$$u(-\infty, \tau) = 0, \quad u(0, \tau) = h_1(\tau) \qquad \text{for } \tau > 0$$

and the initial condition

$$u(x, 0) = h_2(x) \qquad \text{for } -\infty < x < 0.$$

The next step is to convert the two boundary conditions so that they are both zero boundaries (homogenous boundaries). To do this we introduce the transformation $v(x,\tau) = u(x,\tau) - y(x,\tau)$, where $y(x,\tau) = e^x h_1(\tau)$. Using this transformation, the Black-Scholes equation becomes

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + e^x(h_1(\tau) - h_1'(\tau)), \quad \text{for } -\infty < x < 0, \quad \tau > 0. \tag{9}$$

The new, homogenous boundary conditions are

$$v(-\infty,\tau) = 0, \quad v(0,\tau) = 0, \text{ for } \tau > 0$$

and the new initial condition is

$$v(x,0) = h_2(x) - e^x h_1(0), \text{ for } -\infty < x < 0.$$

The transformed continuous autocall PDE problem is thus a standard inhomogeneous PDE problem with homogeneous boundary conditions. By further simplifying notation, the PDE problem can be solved using methods in Evans (2010). Specifically, let $h_3(x) = h_2(x) - e^x h_1(0)$ and $h_4(x,\tau) = e^x(h_1(\tau) - h_1'(\tau))$. The PDE problem is then the following general form

$$\begin{aligned}
&\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + h_4(x,\tau), \quad \text{for } -\infty < x < 0, \quad \tau > 0, \\
&v(-\infty,\tau) = 0, v(0,\tau) = 0, \text{ and } v(x,0) = h_3(x).
\end{aligned}$$

The solution to this PDE is

$$\begin{aligned}
v(x,\tau) \;=\; & \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{0} h_3(s) \left(e^{-(x-s)^2/4\tau} - e^{-(x+s)^2/4\tau}\right) ds + \\
& \int_0^\tau \int_{-\infty}^{0} \frac{h_4(s,r)}{2\sqrt{\pi(\tau-r)}} \left(e^{-(x-s)^2/4(\tau-r)} - e^{-(x+s)^2/4(\tau-r)}\right) ds dr.
\end{aligned} \tag{10}$$

Once $v(x,\tau)$ is solved, we can now solve our transformation $u(x,\tau) = v(x,\tau) + e^x h_1(x)$. Once this function is solved we can fold back and find the value of $V(S,t)$.

3 Example of an Autocallable Structured Product

As an example of our valuation methodology we describe a simple autocallable structured product:[10] If the reference asset has a cumulative positive return on any autocall date, the

[10]This example is similar in its features to one of the more popular brands, the "Autocallable Optimization Securities with Contingent Protection". More than \$1.4 billion in face value of these products were issued in 2009. See the Appendix for more details of the different brands of autocallable structured products and their main features.

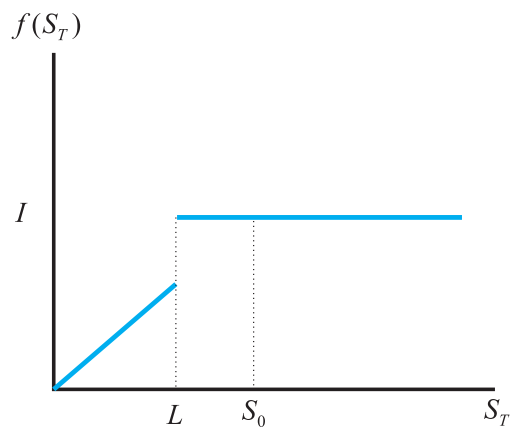
autocallable structured product is called and investors will receive a positive, pre-specified yield. If the product is not called, at maturity the payoff will be:

$$V(S,T) = f(S) = \begin{cases} I = S_0, & S > L; \\ S, & \text{otherwise}, \end{cases} \tag{11}$$

where I is the autocallable structured product's face value, S_0 is the reference asset's initial value, S is the reference asset's final value, and L is the threshold price.

If the autocallable structured product is not called, investors will receive a 0% or a negative. Figure 3 illustrates the autocallable structured product's payoff at maturity if it is not called.

Figure 3: Maturity Payoff if the Autocallable Structured Product is Not Called.



To demonstrate the application of our models, we value three stylized types of autocallable structured products. The first example, our benchmark case, does not have an autocall feature, but has a constant coupon payment. The payoff structure resembles a plain vanilla reverse convertible structured product. The second type has an autocall feature with monthly autocall dates, and the third type has an autocall feature with continuous autocall dates.

For all three examples we assume that the reference asset's initial stock price S_0 and the face value of the note I are both \$100, the call price is \$102, the risk-free rate r is 5%, the volatility σ of the reference asset is 20%, the dividend yield q of the reference asset is 1%, the issuer's CDS spread $C\bar{D}S$ is 1%, the contract length T is one year, and the threshold L is \$80. If the reference asset's price is over the call price on an autocall date (i.e., $S_t \geq C = 102$), the product will be called and will pay a 9.2% annualized return (i.e., $P_t = He^{Bt} = 100e^{0.092t}$).[11] Many autocallable products have a call price identical

[11] This case is our benchmark case, hence we use a 9.2% coupon rate that makes this example first type non-autocallable note a par value note, i.e., principal = \$100.

to the price of the stock ($C = S_0$), however, our assumption $C > S_0$ is without loss of generality.[12]

Case 1: Benchmark - Not Autocallable

In this case, the valuation of autocallable structured product is relatively straightforward. Because the reference asset's final price follows a lognormal distribution

$$S_T = S_0 e^{(r-q-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}W},$$

the value of the structured product is the discounted expected cash flow

$$V(S_0, 0) = e^{-(r+C\bar{D}S)T}\left(\int_0^\infty f(S_T)g(S_T)dS_T + S_0TB\right).$$

where $g()$ is the PDF of S_T. We set the product's issue date value to be \$100.00 per \$100.00 face value by our choice of parameters. As many have shown (see for example Henderson and Pearson (2010)) reverse convertible structured products tend to be overpriced, that is, that they are issued on average at a price that exceeds the present value of their expected future cash-flows. We use this as a benchmark example and hence set it artificially to be priced at face value.

Case 2: Autocallable at Discrete Call Dates

Generally, autocallable structured products have discrete autocall dates. We assume that the product in this example is autocallable monthly.

We first implement the explicit finite difference method to calculate the product value. We set the range of x to be [-5, 0] and the range of τ to be $[0, \frac{T\sigma^2}{2}]$. The resulting $n \times m$ grid has $1000 \times 500 = 500,000$ blocks. Following Equation (6), the value of the product is \$98.39 per \$100.00 face value. We also calculate the monthly probabilities of the autocall being exercised p_i, and show the results in Table 1.

Table 1: The Probability of the Product Being Called on each Monthly Call Date, Conditional on Not Being Called at an Earlier Date.

Month i	1	2	3	4	5	6
p_i	0.3767	0.1435	0.0781	0.0506	0.0361	0.0275
Month i	7	8	9	10	11	12
p_i	0.0218	0.0178	0.0149	0.0127	0.0110	0.0096

[12] In a continuous case, if the call price were identical to the stock price the product would likely be immediately called at issuance, defeating the point of such a call provision.

Case 3: Continuously Autocallable

If the call dates are continuous, we can follow the steps in Section 2.3 to get the closed-form solution.

After the first phase of change of variables, we get a homogeneous heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad \text{for } -\infty < x < 0, \quad \tau > 0,$$
$$u(-\infty,\tau) = 0, \quad u(0,\tau) = C^{-1}e^{-\beta\tau}P_t, \quad u(x,0) = C^{-1}e^{-\alpha x}f(Ce^x),$$

Using our notation, $h_1(\tau) = C^{-1}e^{-\beta\tau}P_t$ and $h_2(x) = C^{-1}e^{-\alpha x}f(Ce^x)$. Applying the second phase of change of variables to make the boundary conditions equal zero. Let $y(x,\tau) = e^x h_1(\tau) = C^{-1}e^{x-\beta\tau}P_t$ and a new function $v(x,\tau) = u(x,\tau) + y(x,\tau)$, then the PDE changes to an inhomogeneous equation

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + C^{-1}e^{x-\beta\tau}P_t(1+\beta+\frac{2B}{\sigma^2}), \quad \text{for } -\infty < x < 0, \quad \tau > 0,$$
$$v(-\infty,\tau) = 0, \quad v(0,\tau) = 0, \quad v(x,0) = C^{-1}e^{-\alpha x}f(Ce^x) - \frac{H}{C}e^{x+BT}.$$

Here $h_3(x) = C^{-1}e^{-\alpha x}f(Ce^x) - \frac{H}{C}e^{x+BT}$ and $h_4(x,\tau) = C^{-1}e^{x-\beta\tau}P_t(1+\beta+\frac{2B}{\sigma^2})$. Applying Equation (10), the solution is

$$\begin{aligned}
v(x,\tau) \;=\; & \frac{S_0}{C}e^{\alpha^2\tau-\alpha x}\left[N\left(\frac{-x+2\alpha\tau}{\sqrt{2\tau}}\right) - N\left(D_1 - \frac{x}{\sqrt{2\tau}}\right)\right] - \\
& \frac{S_0}{C}e^{\alpha^2\tau+\alpha x}\left[N\left(\frac{x+2\alpha\tau}{\sqrt{2\tau}}\right) - N\left(D_1 + \frac{x}{\sqrt{2\tau}}\right)\right] + \\
& e^{D_2+(1-\alpha)x}N\left(D_3 - \frac{x}{\sqrt{2\tau}}\right) - e^{D_2-(1-\alpha)x}N\left(D_3 + \frac{x}{\sqrt{2\tau}}\right) - \\
& \frac{H}{C}e^{BT+x+\tau}N\left(\frac{-x-2\tau}{\sqrt{2\tau}}\right) + \frac{H}{C}e^{BT-x+\tau}N\left(\frac{x-2\tau}{\sqrt{2\tau}}\right) + \\
& \frac{H}{C}(1+\beta+\frac{2B}{\sigma^2})\int_0^\tau e^{D_4+x}N\left(D_5 - \frac{x}{\sqrt{2(\tau-r)}}\right) - \frac{H}{C}e^{D_4-x}N\left(D_5 + \frac{x}{\sqrt{2(\tau-r)}}\right)dr,
\end{aligned}$$

where the parameters are

$$\begin{aligned}
D_1 &= \frac{\log(L/C)+2\alpha\tau}{\sqrt{2\tau}}, \\
D_2 &= \tau(\alpha-1)^2, \\
D_3 &= \frac{\log(L/C)+2\tau(\alpha-1)}{\sqrt{2\tau}}, \\
D_4 &= BT - (\beta + \frac{2B}{\sigma^2} + \tau - r), \\
D_5 &= -\sqrt{2(\tau-r)}.
\end{aligned}$$

The value of the of the product is $V(S,t)$ valued at $(S_0, 0)$, where the form is

$$V(S,t) = Ce^{\alpha x + \beta\tau}(v(x,\tau) + y(x,\tau)).$$

The value $V(S,0)$ is \$99.54.

Comparing the three cases

We can now compare the values in the three different cases: \$100.00, \$98.39, and \$99.54 respectively. Investments with the autocall feature are worth less than their non-autocallable benchmark. The reason for it is fairly intuitive. A non-autocallable investment essentially guarantees a coupon payment until maturity. With an autocall feature, the coupon may be paid for a shorter period or may not be paid at all. Since both investments share the same downside risk, adding the autocall feature (without adjusting the price or the coupon) lowers the value of the investment.

In this example the continuously autocallable structured product is more valuable than the discrete autocallable structured product. Although this is not necessarily always true. Once the coupon payment of a plain vanilla reverse convertible is replaced with an autocallable feature, the investment has a higher value if it is called and the longer it takes to get called. A discrete autocallable feature is less likely to be called, but holding all else equal, may be called later if it is called. Hence, it is more likely for a continuous feature to be more valuable than a discrete one but this does not have to be always the case.

Real-life example

We calculate the product value of a real "Autocallable Optimization Securities with Contingent Protection" note issued by UBS.[13] The note is linked to the stock of Bank of America. It was issued on March 26, 2010 and had a maturity of one year. The reference asset's price on the issue date was $S_0 = \$17.90$. The dividend yield q and implied volatility of the underlying stock σ were 0.2235% and 35.21% respectively. UBS's one year CDS spread was 0.4531%. On the issue date, the one year continuously compounded risk-free rate was 0.4951%. The call price C equaled the initial price S_0. If the note were called, investor would receive a return of 16.1%, and if it were not called, the contingent protection level was $L = 0.7S_0$. Applying our methods, we get a product value of \$97.73 per \$100.00 invested.

4 Conclusion

An Autocallable embedded in a structured product immediately converts the structured product if the reference asset's price exceeds the call price on a call date. The feature has been embedded in many different types of structured products, including Absolute Return Barrier Notes and Optimization Securities with Contingent Protection.

[13]The CUSIP for the product is 90267C136. See the product's pricing supplement at http://www.sec.gov/Archives/edgar/data/1114446/000139340110000136/c178916_690465-424b2.htm

We provide a general partial differential equation framework to model autocallable structured products. We solve the PDE for autocallable structured products with discrete call dates, for which there is typically not a closed-form solution, by using the finite difference method. For continuous autocallables, we derive the closed-form solution. We illustrate our modeling approaches with an example. We then quantify the incremental cost of adding an autocall feature to a plain-vanilla reverse-convertible. We also show the difference between the value of an autocall feature with continuous call dates and one with discrete call dates.

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A Appendix - Descriptions of Select Existing Autocallable Structured Products

A.1 Autocallable Optimization Securities with Contingent Protection

Autocallable Optimization Securities with Contingent Protection, have been issued by several investment banks, including Royal Bank of Canada, UBS, JPMorgan, and HSBC.

Payout if the Product is Called

Autocallable Optimization Securities with Contingent Protection generally have monthly or quarterly call dates, with the final call date being at the product's maturity. In general, the product is called if the reference asset has a positive cumulative return on the call date. When the product is called, investors receive the product's face value plus a pre-specified annual yield.

Payout if the Product is not Called

The payout if not called varies by issuance between yielding a 0% return, or a negative return tied to the stock return of the reference asset.

Most products compare the reference asset's final price to a threshold. If the final price is at or above the threshold, investors receive the product's face value at maturity. If the final price is below the threshold, investors receive the same negative percentage return as the reference asset.

Some products compare the reference asset's *lowest* price during the product's life to a threshold. If the lowest price is at or above the threshold, investors receive the product's face value at maturity. If the lowest price is below the threshold, investors receive the same negative percentage return as the reference asset.

A.2 Autocallable Absolute Return Barrier Notes (Autocallable ARBNs)

Autocallable ARBNs, are continuously callable structured products, issued by Lehman Brothers and UBS. A non-autocallable ARBN is analyzed in depth in Deng et al. (2010b).

Payout if the Product is Called

An autocallable ARBN generally has continuous call dates. The product is called whenever the reference asset's price crosses either an upper or a lower barrier, advancing the return of the structured product's face value.

Payout if the Product is not Called

The note is not called if the reference asset's price stays within the barriers. At maturity, the note pays investors the absolute return of the reference asset, which is a return bounded by the size of the barriers.

A.3 (Semi-) Annual Review Notes with Contingent Principal Protection

Semi-Annual and Annual Review Notes with Contingent Principal Protection, have been issued by several investment banks, including JPMorgan, Credit Suisse, and HSBC.

Payout if the Product is Called

Review Notes with Contingent Principal Protection have discrete call dates. The frequency of the call dates varies, with Annual Review Notes having annual call dates and Semi-Annual Review Notes having semi-annual call dates. In both cases, the final call date is generally at the product's maturity.

In general, the product is called if the reference asset has a positive cumulative return on a call date. However, some products exercise the autocall if the reference asset's cumulative return is positive *or* not too negative (e.g., -10%). Regardless of the autocall trigger, exercising the autocall entitles investors to receive the product's face value plus a pre-specified annual yield.

Payout if the Product is not Called

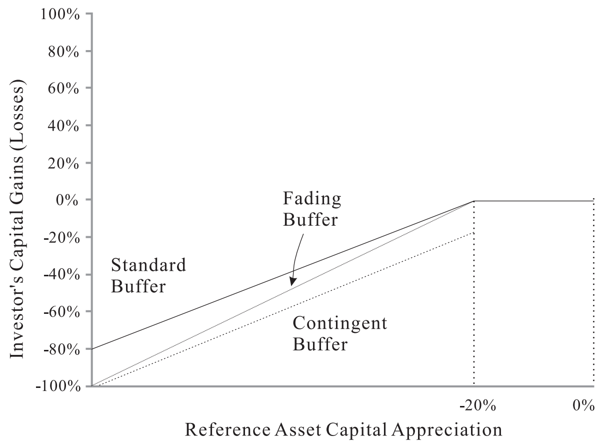
If the product is not called, the payoff at maturity is guaranteed to be no more than the face value of the product. All of the Review Notes we examined have one of three kinds of loss buffers, which we refer to as standard buffers, contingent buffers, and fading buffers. Regardless of the buffer style, investors receive the product's face value at maturity as long as the reference asset's cumulative return is above the buffer (e.g., -20%). If the reference asset's return is below the buffer, investors will lose money.

Review Notes with standard buffers expose investors to any loss in the reference asset beyond the buffer. Thus, a -20% standard buffer will offset a -23% return on the reference asset so the investor only loses 3%.

Review Notes with contingent buffers expose investors to all of the reference asset's losses if the loss is greater than the buffer. For example, if the product has a -20% buffer and the reference asset has a -23% return, investors will receive a -23% return. However, those same investors would receive a 0% return if the reference asset had a -19% return.

Review Notes with fading buffers provide a standard buffer that diminishes as the reference asset's return gets worse. Figure 4 graphs the relationship between the protection offered by a fading buffer and the reference asset's return. The graph shows that the investor begins with a -20% buffer, but the buffer becomes smaller as the reference asset's return becomes more negative. In the extreme, a 0% buffer corresponding to a reference asset return of -100%. Thus, an reference asset return of -23% would equate to a -23.75% loss for the investor.

Figure 4: Standard Buffers, Contingent Buffers, and Fading Buffer



A.4 Autocallable Reverse Convertible Notes

Autocallable Reverse Convertible Notes (issued by Eksportfinans and HSBC) and Autocallable Reverse Exchangeable Notes (issued by JPMorgan) both make coupon payments.

Payout if the Product is Called

Autocallable Reverse Convertible Notes generally have a single, discrete call date early in the life of the product. If the product's reference asset has a positive cumulative return on the call date, the product is called and investors receive any accrued coupon payments and the face value of the note.

Autocallable Reverse Exchangeable Notes are similar, but tend to have multiple discrete call dates or continuous call dates after an initial non-callable period.

Payout if the Product is not Called

If the product is not called, the payout at maturity is similar to a non-autocallable Reverse Exchangeable Note or a Reverse Convertible Note. If the reference asset's price ever crosses a barrier set below its initial price, investors receive the coupon payments plus the product's face value reduced by the lesser of a 0% return or the reference asset's percentage return at maturity. If the reference asset's price never crosses the barrier, investors receive the coupon payments plus the face value of the product.

Some products compare the reference asset's final value, rather than its lowest value, to the barrier return. If the reference asset's final value is below the barrier, the investor is exposed to the reference asset's losses. Otherwise, the product returns its face value. Either way, investors receive the coupons.

A.5 Strategic Accelerated Redemption Securities

Bank of America, Merrill Lynch, and Eksportfinans have all issued Strategic Accelerated Redemption Securities.

Payout if the Product is Called

These autocallable structured products have discrete call dates, with the final call date being at the product's maturity. The product is called if the reference asset has a non-negative return on the call date. When the product is called, investors receive a pre-specified yield on their investment.

Payout if the Product is not Called

If the product is not called, the payout at maturity is similar to a Reverse Exchangeable Note, except that Strategic Accelerated Redemption Securities do not pay coupons. If the reference asset's ending value is not below a threshold, investors receive the face value of the product. If the ending value of the reference asset is below the threshold, investors lose a multiple of the reference asset's negative return. Although the multiple can theoretically be less than 1 or greater than 1, all of the products we saw had a multiple of 1.

Some products do not have a threshold. Instead, investors are guaranteed to lose a multiple of the reference asset's negative return if the product is not called.

A.6 Bear Market Strategic Accelerated Redemption Securities

Bank of America and the Norwegian credit institution Eksportfinans have also issued Bear Market Strategic Accelerated Redemption Securities. These structured products are the same as regular Strategic Accelerated Redemption Securities, except that investors lose money if the reference asset's return is too *high* and earn a pre-specified yield if the reference asset *loses* value.

A.7 Premium Mandatory Callable Equity-Linked Securities (PACERS)

PACERS, issued by Citigroup, pay coupons and have a set of discrete call periods. Each period is a set of two or three continuous call dates. If the reference asset's value on any call date is equal to or greater than its initial value, the product is called and investors receive a pre-specified yield in addition to the accrued coupons.

If the product is not called, the payout is similar to that of an Autocallable Reverse Convertible Note. Specifically, investors receive the same percentage return as the reference asset if the reference asset's ending value is below a threshold (e.g., -25%). Otherwise, investors receive the face value of the note. Regardless of the reference asset's value, investors receive the coupons if the product is not called.

Also similar to Autocallable Reverse Convertible Notes, some PACERS compare the reference asset's lowest value, rather than the final value, to the threshold to determine whether investors receive the face value of the PACERS or the same return as the reference asset.