The Properties of Short Term Investing in Leveraged ETFs

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Abstract

The daily returns on leveraged and inverse-leveraged exchange-traded funds (LETFs) are a multiple of the daily returns of a reference index. Because LETFs rebalance their leverage daily, their holding period returns can deviate substantially from the returns of a leveraged investment. While about half of LETF investors hold their investments for less than a month, the standard analysis of these investments uses a continuous time framework that is not appropriate for analyzing short holding periods, so the true effect of this daily rebalancing has not been properly ascertained.

In this paper, we model tracking errors of LETFs compared to a leveraged investment in discrete time. For a period lasting a month or less, the continuous time model predicts tracking errors to be small. However, we find that in a discrete time model, daily portfolio rebalancing introduces tracking errors that are not captured in the continuous time framework. On average, portfolio rebalancing accounts for approximately 25% of the total tracking error, and in certain scenarios the rebalancing tracking error could rise to as high as 5% in 3 weeks and can dominate the total tracking error. Since investors in LETFs have short average holding periods and high average turnover ratios, the effects of portfolio rebalancing must be accurately accounted for in the analysis of LETF returns.

1 Introduction

Exchange Traded Funds (ETFs) were introduced in the US markets in 1993 and their number has grown rapidly ever since. By the end of January 2011 there were 943 ETFs with combined

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assets of more than $1 trillion.[2] Originally, ETFs tracked broad-market indexes such as the S&P 500 index. More recently, ETFs with more complicated exposures to underlying assets and more complex investment strategies have been issued. For example, the daily return on the leveraged "Ultra S&P 500 ProShares" (SSO) is twice the daily return of the S&P 500 Index. Leveraged and Inverse ETFs (LETFs) were first issued in the United States in June 2006 by the ProFunds Group; there are now more than 400 LETFs with combined assets of more than $120 billion.[3]

Although LETFs are a relatively new type of ETF, there is an emerging literature describing their properties. (Cheng & Madhaven, 2009) and (Avellaneda & Zhang, 2010) establish the properties of LETFs in a continuous time framework. They demonstrate the potentially substantial deviation between the underlying index return and the LETF return due to the daily rebalancing of the LETF. Most notably, they develop the following relationship between the return of an LETF and the return of its underlying index for a long holding period:

$$1 + R_t^A = (1 + R_t^S)^x \exp\left(\frac{x - x^2}{2}\sigma^2 t\right) \quad \textbf{(1)}$$

where x is the leverage size of the LETF, R_t^A and R_t^S are the holding period returns of LETF and the underlying index from time 0 to t, and σ denotes the volatility of the underlying index.[4] As indicated by the equation, holding all else equal, the higher the volatility of the underlying index, the lower the return of an LETF.

(Guedj, et al., 2010) investigate these tracking errors between LETFs and their underlying indices. They illustrate that an LETF could potentially perform much worse than its reference index, especially when realized volatility is high. The same point is also made in (Lu, et al., 2009) which analyzes the long term behavior of LETFs. (Avellaneda & Zhang, 2010) study LETF models in continuous and discrete time. They conclude that in discrete time settings, the return of LETFs are path-dependent, relying on the realized variance in the holding period. They also consider expense ratios and borrowing costs associated with LETFs. However, they assume the effects of daily rebalancing on the overall performance of an LETF

[2] See Investment Company Institute (ICI) - http://www.ici.org/research/stats/etf

[3] Data source (Bloomberg).

[4] This model and all models presented in this paper assume constant volatility.

are small for short holding periods and therefore focus on long holding periods. All these papers highlight how costly it can be for an investor to hold an LETF over a long holding period as the LETF return will likely deviate substantially from the return of a leveraged investment that does not rebalance its portfolio daily.

When held for a short time, the difference between the returns to LETFs and to traditional leveraged investments is limited, which is why LETF issuers recommend using LETFs for only short term trading strategies.[5] However, the effect of discretization errors created by the daily rebalancing of LETF portfolios is not well understood. To achieve the desired leveraged returns of the underlying index, a LETF manager must either borrow or short the same portfolio as the index, typically on a daily basis. In this paper, we investigate the properties and characteristics of investing in an LETF over a short holding period. We model and analyze the return and tracking error of LETFs in a discrete time framework, allowing for a more complete analysis of short holding periods.

First, as a benchmark case, we follow the literature and perform an analysis in continuous time, modeling an LETF's returns and tracking errors. We calculate and compare the expected return and volatility of an LETF to a fixed-initial-leverage investment such as one purchased through a margin account. We analyze the criteria necessary for an LETF to outperform a fixed-initial-leverage investment on the same underlying index. For short holding periods, the probability that fixed-initial-leverage investments outperform LETFs is about 68%. The 68% probability is invariant with changes in the underlying mean, volatility, leverage size, and holding time. When a holding period is short, the returns of LETFs track those of fixed-initial-leverage investments closely, but they deviate substantially when held for a long period of time.

Second, we develop a model of LETF returns in a discrete time setting, i.e., with daily rebalancing (in contrast to continuous time models which only rely on the sample mean of returns). In the discrete time formulation, the holding period returns of LETFs are a function of the sample mean and sample variance of the daily returns of the index. As a consequence, there are additional tracking errors presented in the discrete time setting related to magnitude of the sample variance. When the length of the holding period increases, the discrete time results

[5] See footnote 3

converge to the continuous time results as the sample variance converges to the expected variance. Over short holding periods however, when continuous time models predict minimal tracking error, the extent of tracking errors due to rebalancing can be significant. We quantify this additional tracking error and calculate its magnitude under different scenarios.

The discrete time model is more appropriate than the continuous time model for LETFs because LETFs only rebalance their leverage once a day. Continuous time models effectively assume that rebalancing is performed continuously. This may be a reasonable approximation for long holding periods, but when the holding period is short, the assumption of a continuous rebalancing creates a discrepancy between the modeled returns and likely realized returns. Using a discrete time setting, we model daily rebalancing and obtain more accurate predictions of the distribution of LETF returns for short holding periods.

Based on our analysis of LETFs, we find that almost 50% of the existing funds have an average holding period of less than a month. We calculate the holding periods for each LETF when tracking errors between discrete and continuous returns are large. The large discrepancies occur when the realized volatility is substantially greater than the expected volatility. We explain several other reasons that possibly explain the large discrepancies, including the imperfectness in leverage and stochastic volatilities over time.

2 Benchmark Case - Continuous Time Model

We start by developing a continuous time analysis of LETF returns and properties similar to the one developed in (Cheng & Madhaven, 2009). We then develop the discrete time model and compare the results to the continuous time models for short holding periods. In contrast to (Cheng & Madhaven, 2009), we concentrate our analysis on the comparison of the LETF properties with those of a fixed-initial-leveraged investment. A fixed-initial-leveraged investment earns a fixed multiple of the holding period return on an index and is achieved by borrowing and investing in an ETF, so for simplicity we will refer to this strategy as an METF.[6]

[6] As our focus is on holding periods measured in days not months and the cross-sectional variation in holding period returns, we ignore borrowing costs.

2.1 Model

We perform the continuous time analysis on three stochastic processes: the levels of the underlying index (S_t), the LETF (A_t) and the METF (M_t). Without loss of generality, we simply set the starting levels of all three processes to be the same $S_0 = A_0 = M_0$. In the following analysis we explore evolutions among these three processes.

Suppose the underlying index level follows a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{2}$$

where μ and σ are the expected mean and volatility of the return. When set in a risk-neutral world, $\mu = r - q$ where q is a dividend yield and r is a risk-free rate (Hull, 2008). An LETF with a leverage x also follows a geometric Brownian motion as

$$\frac{dA_t}{A_t} = x\frac{dS_t}{S_t} = x\mu dt + x\sigma dW_t \tag{3}$$

This implies that the LETF leverages up the index return x times and volatility $|x|$ times.

Since the margin account leverages up the holding period return of the index, at any time t, the level of an METF M_t satisfies the following relation with the underlying index:

$$\frac{M_t}{M_0} = \left(\frac{S_t}{S_0} - 1\right) x + 1 = x\frac{S_t}{S_0} - (x - 1) \tag{4}$$

Applying Itô's Lemma to $M_t(S_t, t)$ we obtain a stochastic partial differential equation for M_t:

$$\begin{aligned} dM_t &= x\mu S_t dt + x\sigma S_t dZ_t \\ &= \mu(M_t + (x-1)M_0)dt + \sigma(M_t + (x-1)M_0)dZ_t \end{aligned} \tag{5}$$

Note that the equation has the term M_0 which implies that the value of an METF, unlike S_t and A_t, is dependent on the initial value at the starting time $t = 0$.

Since both S_t and A_t follow a geometric Brownian motion, at any given time t they are log-normally distributed:

$$S_t = S_0 \exp\left((\mu - \sigma^2/2)t + \sigma\sqrt{t}Z\right)$$

$$A_t = A_0 \exp\left((x\mu - x^2\sigma^2/2)t + x\sigma\sqrt{t}Z\right) \tag{6}$$

where Z is a standard normal variable. For METFs, instead of solving the partial differential Equation (5), we use the mapping function (4) to get the functional form for M_t at any time. We summarize the mean and standard deviation of the holding period returns R_t^S, R_t^A and R_t^M in Table 1.[7]

Table 1: Summary of the Holding Period Returns

	Mean	Standard Deviation
Index Return R_t^S	$e^{\mu t} - 1$	$\sqrt{e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)}$
LETF Return R_t^A	$e^{x\mu t} - 1$	$\sqrt{e^{2x\mu t}\left(e^{x^2\sigma^2 t} - 1\right)}$
METF Return R_t^M	$x(e^{\mu t} - 1)$	$\lvert x\rvert\sqrt{e^{2\mu t}\left(e^{\sigma^2 t} - 1\right)}$

Because the levels of the LETF A_t and index S_t share the same sample path, the quantity Z should be identical for S_t and A_t in Equation (6). Cancelling the variable Z in the equation results the relationship between A_t and S_t previously given in Section 1 (Avellaneda & Zhang, 2010) and (Cheng & Madhaven, 2009)

$$\frac{A_t}{A_0} = 1 + R_t^A = \left(\frac{S_t}{S_0}\right)^x \exp\left(\frac{x - x^2}{2}\sigma^2 t\right) \qquad \textbf{(7)}$$
$$= (1 + R_t^S)^x \exp\left(\frac{x - x^2}{2}\sigma^2 t\right)$$

We prove this equation using Itô's Lemma. We define a new process G_t which is the ratio of $\frac{A_t}{A_0}$ and $\left(\frac{S_t}{S_0}\right)^x$:

$$G_t(A_t, S_t, t) = \frac{\frac{A_t}{A_0}}{\left(\frac{S_t}{S_0}\right)^x}$$

Applying Itô's Lemma on the process $G_t = (A_t, S_t, t)$:

[7] The holding period returns are defined as $R_t^S = \frac{S_t}{S_0} - 1$, $R_t^A = \frac{A_t}{A_0} - 1$ and $R_t^M = \frac{M_t}{M_0} - 1$. According to the payoff property of the METF, its holding period return satisfies $R_t^M = xR_t^S$.

$$
\begin{aligned}
dG_t &= \left(\frac{\partial G_t}{\partial S_t}\mu S_t + \frac{\partial G_t}{\partial t} + \frac{\partial G_t}{\partial A_t}x\mu A_t + \frac{1}{2}\frac{\partial^2 G_t}{\partial S_t^2}(\sigma S_t)^2 + \frac{1}{2}\frac{\partial^2 G_t}{\partial A_t^2}(x\sigma A_t)^2\right. \\
&\qquad \left. + \frac{\partial^2 G_t}{\partial S_t A_t}(\sigma S_t)(\sigma A_t)\right)dt + \left(\frac{\partial G_t}{\partial S_t}\sigma S_t + \frac{\partial G_t}{\partial A_t}x\sigma A_t\right)dZ_t \\
&= \frac{1}{2}(x - x^2)\frac{A_t}{A_0}\left(\frac{S_t}{S_0}\right)^{-x}\sigma^2 dt = \frac{1}{2}(x - x^2)G_t\sigma^2 dt
\end{aligned} \tag{8}
$$

Note that the dZ_t term is canceled in the calculations, thus the stochastic partial differential equation becomes a static partial differential equation, which has a solution

$$
G_t = \exp\left(\frac{x - x^2}{2}\sigma^2 t\right)G_0
$$

This completes the proof of Equation (7).

Combining Equations (4) and (7), the relationship between A_t and M_t follows

$$
\begin{aligned}
\frac{A_t}{A_0} = 1 + R_t^A &= \left(\frac{M_t/M_0 + x - 1}{x}\right)^x \exp\left(\frac{x - x^2}{2}\sigma^2 t\right) \\
&= \left(\frac{R_t^M + x}{x}\right)^x \exp\left(\frac{x - x^2}{2}\sigma^2 t\right)
\end{aligned} \tag{9}
$$

The tracking error between LETF and METF is a process defined as

$$
\begin{aligned}
Tracking\ Err = M_t - A_t &= S_0(R_t^M - R_t^A) \\
&= S_0\left(xR_t^S - (1 + R_t^S)^x \exp\left(\frac{x - x^2}{2}\sigma^2 t\right) + 1\right)
\end{aligned} \tag{10}
$$

At given time t, the expected tracking error is

$$
\mathbb{E}[Tracking\ Err] = S_0(xe^{\mu t} - e^{x\mu t} - x + 1)
$$

This quantity is small when time t is small, while decreasing to negative infinity as t increases.

2.2 Model Implications

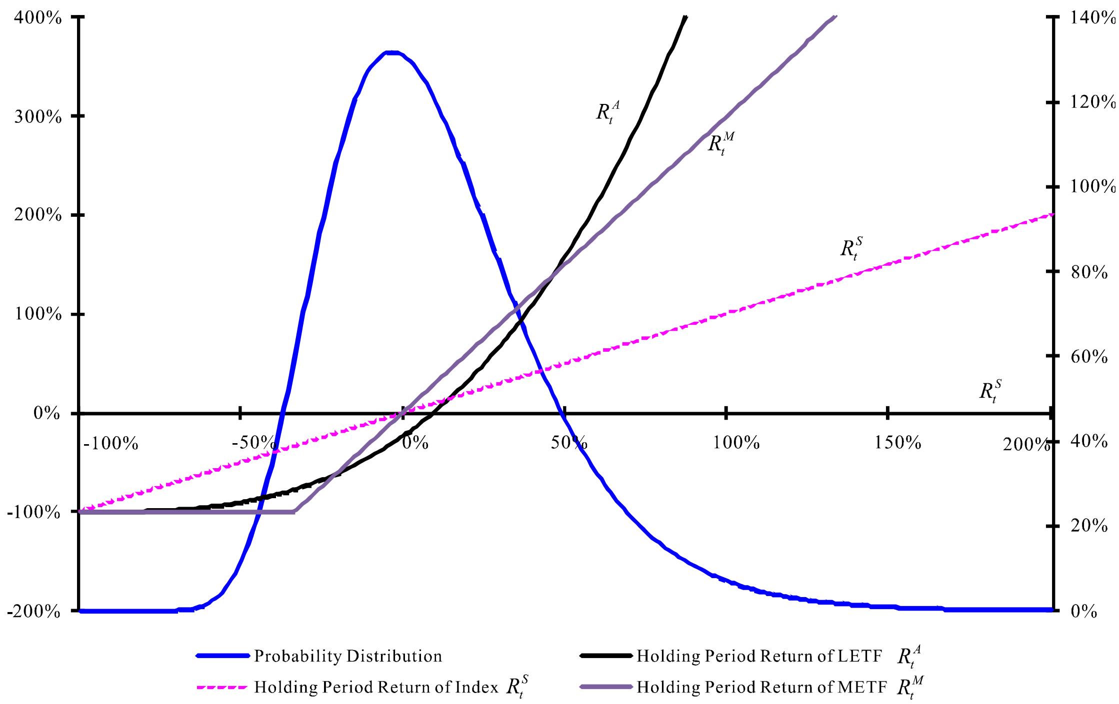
There are several important implications about LETF returns in general and for shorter holding period in particular to discuss before developing the discrete-time model.

2.2.1 Long-Term Returns

First, from Equations (4) and (7), the levels A_t and M_t are mapped in a one-to-one relationship to the underlying index S_t.[8] That is, the values of the LETF and the METF at time t depend on the value of the index at t, S_t, but not on the path over time to t. This mapping holds in a continuous time framework but not in a discrete-time framework.

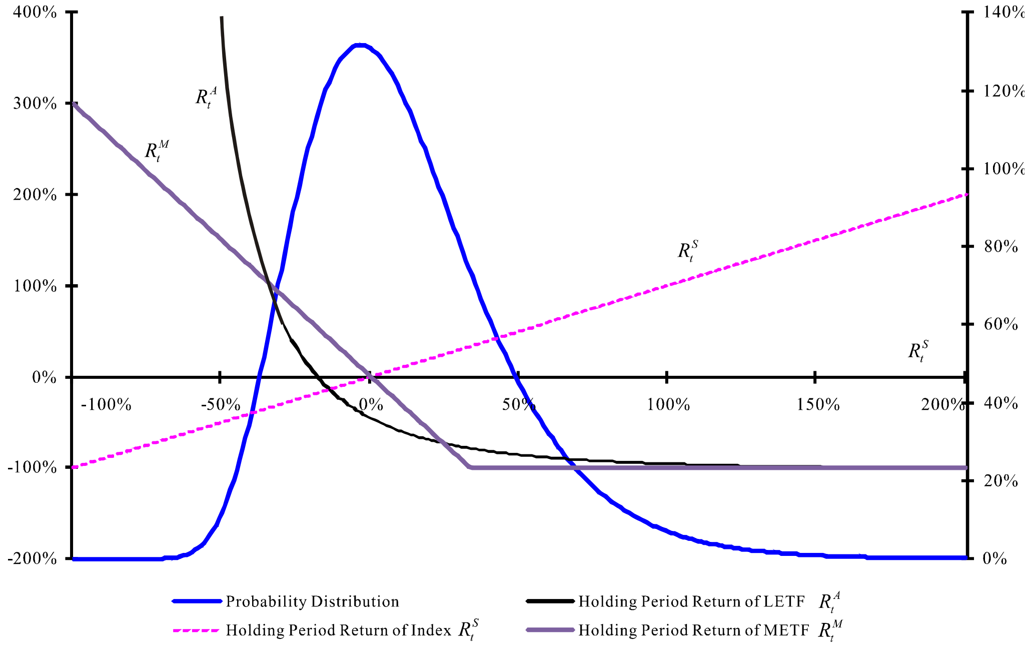
In Figure 1 we plot a comparison of a one year return for an LETF (R_t^A) and an METF (R_t^M) compared to the return of the underlying index (R_t^S). We give two examples: leverage $x = 3$ (in Panel (a)) and $x = -3$ (in Panel (b)). In the same figure, we also plot on the second axis the probability density function for the underlying index (R_t^S), which follows a log-normal distribution.

Figure 1: The plot of holding period returns of LETF R_t^A and METF R_t^M versus R_t^S assuming $\mu = 10\%$, $\sigma = 30\%$, $t = 1$ year and leverage $x = 3$. Since the index holding period return $1 + R_t^S = \frac{S_t}{S_0}$ follows a log-normal distribution, we also plot the probability density function of R_t^S as a reference (right axis)



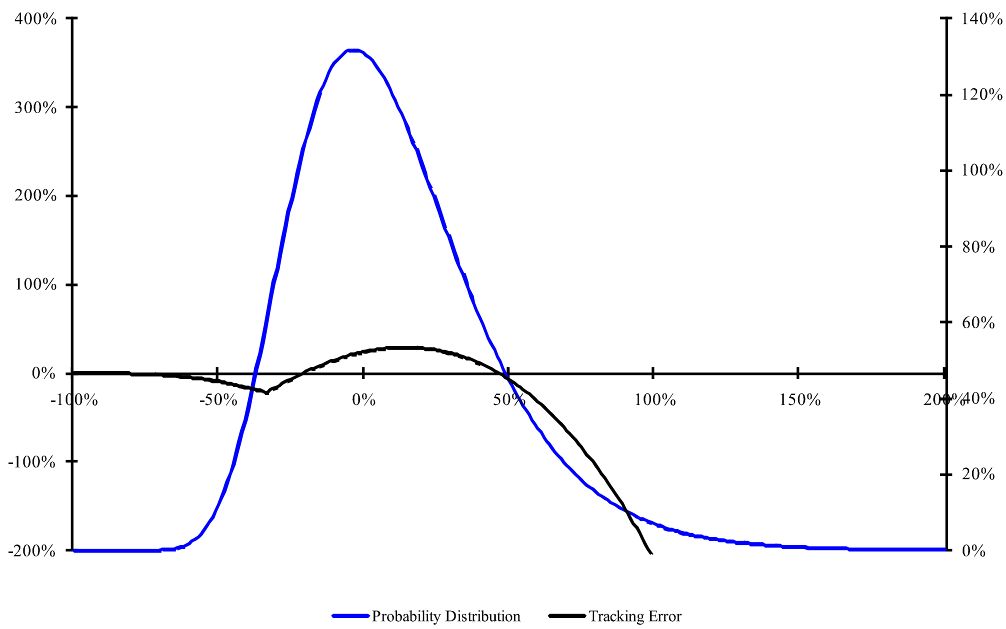
[8] Or equivalently, the holding period returns R_t^A and R_t^M are one-to-one mapped to the index return R_t^S.

(a) Leverage $x = 3$



(b) Leverage $x = -3$

Figure 2: For the Leverage $x = 3$ case, in addition to Figure 1 (a), the tracking error $R_t^M - R_t^A$ is plotted.



First, note that in Panel (a) for index returns between -20.6% and 46.2% the return of an METF dominates that of the LETF. The superimposed probability density highlights the fact that the majority of the probability likelihood is in the range over which the METF dominates the LETF, making a long term investment in an LETF substantially worse than an METF. This is clear from Equation (7): for long holding periods (in this example a year) an LETF will perform worse than a fixed-initial-leveraged investment. Moreover, if the underlying index had a small positive return the LETF (with positive leverage) can under-perform not only the METF but also the underlying index, due to the cost associated with the daily rebalancing of the leverage. Similar behavior holds with the negative leveraged LETF. The METF will suffer 100% losses after an accumulated index loss of -33% for a positive 3x leverage (and 33% for a negative 3x leverage). In contrast, an LETF is never fully wiped out as it rebalances its leverage daily and will converge to a value of zero but will never reach it.

Second, the term $\exp\left(\frac{x-x^2}{2}\sigma^2 t\right)$ in Equation (7) is always less than one, regardless of the leverage size (which is enumerated in the set $\{-3,-2,-1,2,3\}$). This implies that when S_t equals S_0, or when the holding period return of the index and the margin account are both zero ($R_t^S = R_t^M = 0$), the holding period return of the LETF is always negative and underperforms the index and METF. This implication is reflected in Figure 2 which directly plots the tracking error of these two accounts $R_t^M - R_t^A$ versus the realized holding period return R_t^S. The tracking error line crosses the x-axis at the same points regardless of the leverage size.

On the other hand, when the holding period return of R_t^S is significantly positive (increasing to infinity) or negative (decreasing to -100%), the LETF returns exceed the return of the index, see Figure 1. As t increases, the term $\exp\left(\frac{x-x^2}{2}\sigma^2 t\right)$ decreases to zero, which implies that $1 + R_t^A$ becomes significantly less than $(1 + R_t^S)^x$ but as Figure 1 illustrates, this does not imply it is worse than the index value $1 + R_t^S$.

2.2.2 Short-Term Returns

Using a Taylor expansion of the right hand side of Equation (7), we can see that when t is small, the return curves of LETF and METF intersect at two points, which are approximated by $R_t^S \sim \pm\sigma\sqrt{t}$. Since these techniques are used extensively in later sections of the paper, we restate the derivation here.

$$R_t^A = (1+R_t^S)^x \exp\left(\frac{x-x^2}{2}\sigma^2 t\right) - 1$$

$$= \left(1 + xR_t^S + \frac{x(x-1)(R_t^S)^2}{2!} + \cdots\right) \cdot \left(1 + \frac{x-x^2}{2}\sigma^2 t + \frac{\left(\frac{x-x^2}{2}\sigma^2 t\right)^2}{2!} + \cdots\right) - 1$$

$$= xR_t^S + \frac{x(x-1)}{2}((R_t^S)^2 - \sigma^2 t) + O((R_t^S)^3) + O(t^2)$$

The second line of this result is derived from a Taylor expansion of two functions $f(x) = (1+a)^x$ and $f(x) = \exp(x)$.[9] The omitted terms in the series are in the order of $O((R_t^S)^3) + O(t^2)$. When R_t^S and $\frac{x-x^2}{2}\sigma^2 t$ are sufficiently small (or when t is sufficiently small), the following result holds

$$R_t^A \sim xR_t^S + \frac{x(x-1)}{2}((R_t^S)^2 - \sigma^2 t) \qquad \textbf{(11)}$$

Recall that the holding period return of the METF follows $R_t^M = xR_t^S$. The tracking error between the METF and LETF returns (from Equation (10)) is further simplified as

$$Tracking\ Err = R_t^M - R_t^A \sim -\frac{x(x-1)}{2}((R_t^S)^2 - \sigma^2 t)$$

This expression implies that the LETF outperforms the METF when $R_t^S > \sigma\sqrt{t}$ or $R_t^S < -\sigma\sqrt{t}$. When holding both LETF and METF for a short time, i.e., t less than a week or a month, the result (11) holds with a high degree of accuracy. The locations of the intersection points $(\pm\sigma\sqrt{t})$ are independent of the leverage size x.

In addition, since $\frac{S_t}{S_0} = 1 + R_t^S$ follows the lognormal distribution

$$S_t \sim LogNormal\left((\mu - \sigma^2/2)t, \sigma\sqrt{t}\right)$$

we calculate the probability that the METF outperforms the LETF as

$$Prob(R_t^A < R_t^M)$$
$$\sim Prob(-\sigma\sqrt{t} < R_t^S < \sigma\sqrt{t})$$
$$\sim Prob(-\sigma\sqrt{t} < log(1+R_t^S) < \sigma\sqrt{t})$$

[9] The Taylor Series expansion results are $f(a) = (1+a)^x = 1 + xa + x(x-1)a^2/2 + \cdots$ and $f(x) = \exp(x) = 1 + x + x^2/2! + x^3/3! + \cdots$

$$= Prob\left(-\sigma\sqrt{t} < (Z + (\mu - \sigma^2/2)t)\sigma\sqrt{t} < \sigma\sqrt{t}\right)$$
$$= Prob(-1 - (\mu - \sigma^2/2)t < Z < 1 - (\mu - \sigma^2/2)t)$$
$$\sim Prob(-1 < Z < 1)$$
$$= 68.27\%$$

where Z is a standard normal variable. The second line of the approximation re-applies the Taylor expansion of the function $y = \log(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \cdots$. Since $(1+R_t^S)$ follows a lognormal distribution, $\log(1+R_t^S)$ follows a normal distribution $N\left((\mu-\sigma^2/2)t, \sigma\sqrt{t}\right)$. When t is sufficiently small, the term $1-(\mu-\sigma^2/2)t$ is approximately 1.

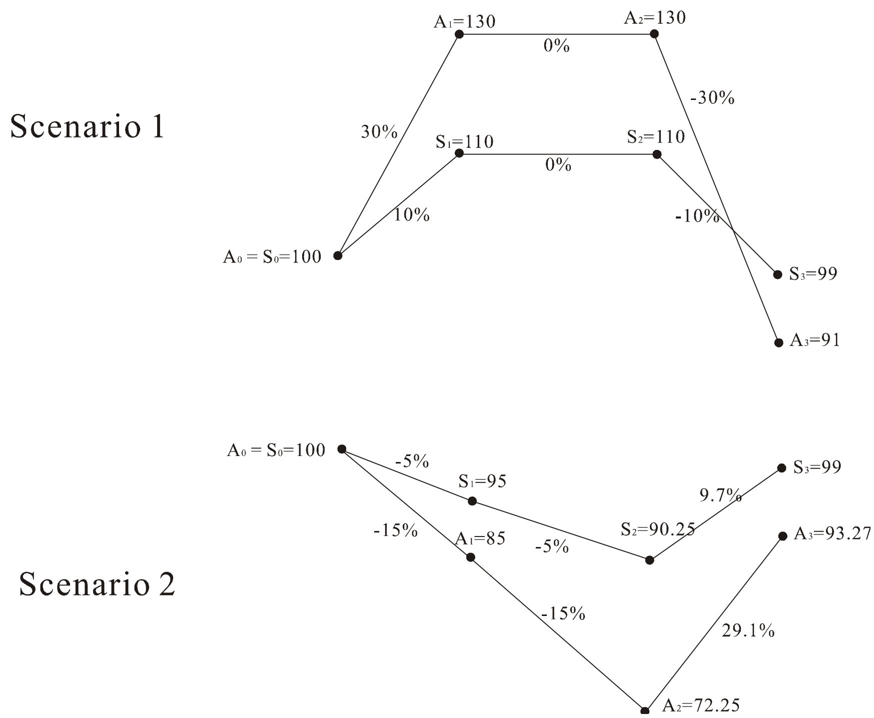
The probability that the METF value exceeds the LETF value is very high when t is small. For example, using the parameter assumptions in Figure 1 (a) $\mu = 10\%$, $\sigma = 30\%$, and leverage $x = 3$, but setting $t = 0.01$, the tracking error equals zero at $R_t^S = 3.12\%$ and $R_t^S = -2.88\%$, which are close to $\pm\sigma\sqrt{t} = \pm 3.00\%$. The probability that the LETF underperforms the METF is 68.27%. We also observe that this probability value is relatively independent of the values of μ and σ. If $x = -3$ as in the Figure 1 (b) case, the two intersection points are $R_t^S = 2.94\%$ and $R_t^S = -3.06\%$. This implies that the result is also quite independent of the amount of leverage.

For long holding periods, the probability that the METF value exceeds the LETF value loses some accuracy but it is still a good approximation. For example, using the above parameters but changing the time $t = 1$, the tracking error curve intersects zero at $R_t^S = 46.2\%$ and $R_t^S = -20.6\%$ which shift significantly from the reference value of $\pm\sigma\sqrt{t} = \pm 30\%$. Therefore, based on our calculations, the value of the METF exceeds the value of the LETF with a true probability of 69.01%.

3 Discrete Time Analysis

We start with a simple example to illustrate the different tracking errors yielded by our continuous time and discrete time models. The example consists of two scenarios for simple three-day investments. See Figure 3 for an illustration. In Scenario 1, the daily returns of the index are 10%, 0%, and -10%. The corresponding LETF with a leverage ratio of 3 has daily returns of 30%, 0% and -30%. If the initial values of both accounts are $A_0 = S_0 = 100$, then the index has an ending value of 99 and the LETF ends with a value 91. The METF has an ending value 97 so the tracking error is -6. In Scenario 2, the index returns are -5%, -5%, and 9.7% for three consecutive days, resulting in a final index value of 99. Though we have the same index value and METF return as in Scenario 1, we observe that the LETF now has an ending level of 93.27 and the tracking error is -3.73.

Figure 3: A simple example of two realized scenarios



Though the ending levels of the index are the same for both scenarios, the ending levels of the LETF and the LETF's tracking error differ. This contradicts the continuous time model result, which states that the level A_t is one-to-one function of S_t regardless of paths. We may conclude that LETFs in discrete time are path dependent and therefore should not be analyzed with continuous time models.

3.1 Model

In the discrete time setting, we assume the discrete time periods are $t_0, t_1, t_2 \dots, t_n$. The time points are equally spaced with a step size $\Delta t = t_i - t_{i-1}, i = 1, \dots, n$, where Δt typically represents the portfolio rebalance frequency. If daily rebalancing is assumed, we may set $\Delta t = \frac{1}{252}$ to represent the length of one day. The *simple return* of the index in each time step is denoted as r_i, where r_i follows a normal distribution with annualized expected return μ and expected volatility σ:

$$r_i \sim N\left(\mu\Delta t, \sigma\sqrt{\Delta t}\right)$$

r_i are also i.i.d. normal random variables.

The index level S_n at time t_n is an accrual of daily returns $S_n = S_0 \prod_{i=1}^{n}(1 + r_i)$. Because LETFs rebalance their portfolios once every time increment $\left[t_{i,} t_{i+1}\right]$, the simple return of the LETF is x times the simple return of the index:

$$r_i^A = x r_i \tag{12}$$

This discrete time equation corresponds to Equation (3) in the continuous time setting. The level of the LETF at time t_n is therefore $A_n = A_0 \prod_{i=1}^{n}\left(1 + r_i^A\right)$.

We first analyze the relationship between A_n and S_n. Introducing a similar ratio as in the continuous time $G_1 = \frac{A_i}{S_i^x}$, on day zero $G_0 = \frac{A_0}{S_0^x}$. On day one,

$$G_1 = \frac{A_1}{S_1^x} = G_0 \frac{1 + r_1^A}{(1 + r_1)^x} = G_0 \frac{1 + x r_1}{(1 + r_1)^x}$$

If we take the log on both sides $\log(G_1) = \log(G_0) + \log(1 + x r_1) - x\log(1 + r_1)$ and use the Taylor expansion result on function $\log(1 + x)$, we have

$$\log(G_1) = \log(G_0) + \frac{x - x^2}{2} r_1^2 + \cdots$$

The omitted terms are in the order of r_1^3. We use backward induction to calculate G_{i+1} from G_i as

$$G_{i+1} = \frac{A_{i+1}}{S_{i+1}^x} = G_i \frac{1 + r_{i+1}^A}{(1 + r_{i+1})^x}$$

Thus, we may derive the general expression

$$\log(G_i) \sim \log(G_0) + \frac{x - x^2}{2} \sum_{k=1}^{i} r_k^2$$

Simplifying the above equation and plugging A_i and S_i back in, as of time t_n we have

$$\frac{A_n}{A_0} \sim \left(\frac{S_n}{S_0}\right)^x \exp\left(\frac{x - x^2}{2} \sum_{i=1}^{n} r_i^2\right) \tag{13}$$

Equation (13) is the discrete time result based on simple returns r_i. Compared to its counterpart in the continuous time setting (Equation (7)), we note the difference is that the term $\sigma^2 t$ in Equation (7) is replaced by $\sum_{i=1}^{n} r_i^2$. However, the new term is not easy to solve quantitatively. We need to further transform the equation to *continuously compounded daily returns.*

Denoting $\hat{r}_i$ as the continuously compounded daily return, the index levels $S_n = S_0 \exp(\sum_{i=1}^{n} \hat{r}_i)$. The translation from the continuously compounded LETF return $\hat{r}_i^A$ to $\hat{r}_i$ is $\exp(\hat{r}_i^A) - 1 = \hat{r}_i^A = x r_i = x(\exp(\hat{r}_i) - 1)$. When Δt is small, i.e., a day, $\hat{r}_i$ is approximately distributed as

$$\hat{r}_i \sim N\left((\mu - \sigma^2/2)\Delta t, \sigma\sqrt{\Delta t}\right)$$

The G_i expressed in terms of $\hat{r}_i$ have similar derivations

$$\begin{aligned} \log(G_1) &= \log(G_0) + \hat{r}_1^A - x\hat{r}_1 \\ &= \log(G_0) + \log(1 + r_1^A) - x\hat{r}_1 \\ &= \log(G_0) + \log(1 + x r_1) - x\hat{r}_1 \\ &= \log(G_0) + \log\big(1 + x(\exp(\hat{r}_1) - 1)\big) - x\hat{r}_1 \\ &= \log(G_0) + \frac{x - x^2}{2}\hat{r}_1^2 + \cdots \end{aligned}$$

The general expression in terms of continuously compounded returns is

$$\frac{A_n}{A_0} \sim \left(\frac{S_n}{S_0}\right)^x \exp\left(\frac{x - x^2}{2}\sum_{i=1}^{n} \hat{r}_i^2\right) \qquad \textbf{(14)}$$

This equation has the same form as the simple return case (13) except using $\hat{r}_i$ instead of r_i. Our analysis is similar to (Avellaneda & Zhang, 2010) in the sense that they also perform a discrete analysis. However, their focus on path dependency leads to a different decomposition (using realized variance instead of sum of mean squared) which does not allow for the same analysis of the tracking error. Using variance is not an optimal method for measuring the tracking error as the decomposition adds an error term of magnitude $nO((\Delta t)^2)$. In fact, many papers on discrete variance modeling directly define ‘realized variance’ as the sum of returns without adjusting the mean term. See for example, (Itkin & Carr, 2010).

The holding period return of the METF $R_{t_n}^M$ is still one-to-one mapped to holding period returns of the index $R_{t_n}^S$ in the discrete time setting. The mean and standard deviation of METFs remain the same as in the continuous time setting (see Table 1). When $n = 1$, the tracking error is zero because the returns of LETF and METF are equal to the leverage of the index return by x times. For a term of t_n, the tracking error in the discrete time setting is calculated as

$$\begin{aligned} Tracking\ Err &= S_0\left(R_{t_n}^M - R_{t_n}^A\right) \\ &= S_0\left(xR_{t_n}^S - \left(1 + R_{t_n}^S\right)^x \exp\left(\frac{x - x^2}{2}\sum_{i=1}^{n} \hat{r}_i^2\right) + 1\right) \\ &= S_0\left(x \exp\left(\sum_{i=1}^{n} \hat{r}_i\right) - \left(1 + \exp\left(\sum_{i=1}^{n} \hat{r}_i\right)\right)^x \exp\left(\frac{x - x^2}{2}\sum_{i=1}^{n} \hat{r}_i^2\right) + 1\right) \end{aligned}$$

3.2 Model Implications

3.2.1 Short-Term Returns

The holding period return of LETF $R_{t_n}^A$ and METF $R_{t_n}^M$ can be expressed in terms of sample mean and sample variance, as can the tracking error. Recall the definition of the sample mean $\overline{\mu}(\hat{r}_i) = \frac{1}{n}\sum_{i=1}^{n} \hat{r}_i$, and sample variance $s^2(\hat{r}_i) = \frac{1}{n-1}\sum_{i=1}^{n} \hat{r}_i^2 - \frac{1}{n^2-n}\left(\sum_{i=1}^{n} \hat{r}_i\right)^2$.

$$\begin{aligned} \frac{A_n}{A_0} &\sim \exp\left(xn\overline{\mu}(\hat{r}_i)\right)\exp\left(\frac{x - x^2}{2}\left((n-1)s^2(\hat{r}_i) + n\overline{\mu}(\hat{r}_i)^2\right)\right) \\ &= \exp\left(xn\overline{\mu}(\hat{r}_i) + \frac{x - x^2}{2}\left((n-1)s^2(\hat{r}_i) + n\overline{\mu}(\hat{r}_i)^2\right)\right) \end{aligned} \tag{15}$$

For METF, using the fact that $S_n = S_0\exp\left(n\overline{\mu}(\hat{r}_i)\right)$ the return is solely a function of sample mean

$$\frac{M_n}{M_0} = 1 + x\left(\exp\left(n\overline{\mu}(\hat{r}_i)\right) - 1\right)$$

Let's review some properties about these two statistics: sample mean and sample variance (Casella & Berger, 2001). They are both random variables, thus A_n is also a random variable. Since $\hat{r}_i$ are i.i.d. normally variables, the sample mean $\overline{\mu}(\hat{r}_i)$ is also normally distributed

$$\overline{\mu}(\hat{r}_i) \sim N\left((\overline{\mu} - \sigma^2/2)\Delta t, \sigma\sqrt{\frac{\Delta t}{n}}\right)$$

The sample variance follows a $\mathcal{X}_{n-1}^2$ distribution with $n - 1$ degrees of freedom

$$\frac{(n-1)s^2(\hat{r}_i)}{\sigma^2\Delta t} \sim \mathcal{X}_{n-1}^2$$

In addition, sample mean and sample variance are *independent* of each other.

In Equation (15), when Δt is small, the value $(n-1)s^2(\hat{r}_i)$ dominates $n\bar{\mu}(\hat{r}_i)^2$, thus the term $n\bar{\mu}(\hat{r}_i)^2$ can be ignored. Moreover, the sample variance $s^2(\hat{r}_i)$ is known as an unbiased and consistent estimator of $\sigma^2\Delta t$. For long holding periods, as n increases, the sample variance converges to $\sigma^2\Delta t$:

$$(n-1)s^2(\hat{r}_i) + n\overline{\mu}(\hat{r}_i)^2 \sim (n-1)s^2(\hat{r}_i) \xrightarrow{n\to\infty} \frac{n-1}{n}\sigma^2 t_n \sim \sigma^2 t_n$$

which makes (15) converge to the continuous time result (7).

A key difference between the discrete and continuous time analyses is that for the LETF, one-to-one mapping between $R_{t_n}^A$ and $R_{t_n}^S$ no longer holds, and nether does the mapping between $R_{t_n}^A$ and $R_{t_n}^M$. This indicates that there are extra tracking errors in the discrete time setting.

We start our analysis of these errors by evaluating the mean and standard deviation of $R_{t_n}^A$.

$$\begin{aligned}
\mathbb{E}\left[R_{t_n}^A\right] &\sim \mathbb{E}\left[\exp\left(xn\overline{\mu}(\hat{r}_i) + \frac{x-x^2}{2}\left((n-1)s^2(\hat{r}_i) + n\overline{\mu}(\hat{r}_i)^2\right)\right) - 1\right] \\
&= \mathbb{E}\left[\exp\left(xn\overline{\mu}(\hat{r}_i) + \frac{x-x^2}{2}n\overline{\mu}(\hat{r}_i)^2\right)\right] \cdot \mathbb{E}\left[\exp\left(\frac{x-x^2}{2}(n-1)s^2(\hat{r}_i)\right)\right] - 1 \\
&= \frac{\exp\left(-\frac{t_n x(\Delta t(-4\mu^2 + 4\mu^2 x + 4\mu\sigma^2 - 4\mu\sigma^2 x - \sigma^4 + \sigma^4 x) - 4x\sigma^2 - 8\mu + 4\sigma^2)}{8(-x\sigma^2\Delta t + x^2\sigma^2\Delta t + 1)}\right)}{\sqrt{-x\sigma^2\Delta t + x^2\sigma^2\Delta t + 1}} \\
&\quad \cdot (1-(x-x^2)\sigma^2\Delta t)^{-\frac{n-1}{2}} - 1 \\
&\sim \exp(x\mu t_n)\exp\left(-\frac{x-x^2}{2}\sigma^2 t_n\right)(1-(x-x^2)\sigma^2\Delta t)^{-\frac{n-1}{2}} - 1
\end{aligned} \tag{16}$$

In Line 2, the expectations are separable because the sample mean and sample variance are independent. The last approximation is due to omitting small terms with Δt, which comes from the inclusion of $\frac{x-x^2}{2}n\overline{\mu}(\hat{r}_i)^2$ in Line 2. In a long term scenario, the additional tracking error vanishes. As t_n increases to infinity, the quantity $(1-(x-x^2)\sigma^2\Delta t)^{-\frac{n-1}{2}}$ converges to

$\exp\left(\frac{x-x^2}{2}\sigma^2 t_n\right)$. $\mathbb{E}\left[R_{t_n}^A\right]$ becomes $\exp(x\mu t_n) - 1$ which matches the continuous time results in Table 1.

The standard deviation of $R_{t_n}^A$ follows a similar derivation

$$\begin{aligned} \operatorname{std}\left(R_{t_n}^A\right) &\sim \operatorname{std}\left(\exp\left(xn\overline{\mu}(\hat{r}_i)\right)\cdot\exp\left(\frac{x-x^2}{2}(n-1)s^2(\hat{r}_i)\right)\right) \\ &= \exp\left(x\mu\mathrm{t_n} - \frac{x-x^2}{2}\sigma^2 t_n\right)\Big(\exp(x^2\sigma^2 t_n)(1-2(x-x^2)\sigma^2\Delta t)^{-\frac{n-1}{2}} \\ &\qquad - (1-(x-x^2)\sigma^2\Delta t)^{-(n-1)}\Big)^{\frac{1}{2}} \end{aligned} \tag{17}$$

Again, as $\mathrm{t_n}$ increases to infinity, $\operatorname{std}\left(R_{t_n}^A\right)$ converges to continuous time result $\sqrt{\exp(2x\mu t_n)(\exp(x^2\sigma^2\mathrm{t_n})-1)}$ as in Table 1.

We have shown that for short holding periods, the overall mean and standard deviation of the holding period return $R_{t_n}^A$ is only slightly biased from the continuous time case. As t_n increases, the difference vanishes. For ease of derivation, we use the following notations in the remainder of the text.

$$A = (1-(x-x^2)\sigma^2\Delta t)^{-\frac{n-1}{2}}$$

$$B = (1-2(x-x^2)\sigma^2\Delta t)^{-\frac{n-1}{2}}$$

and

$$C = \exp\left(\frac{x-x^2}{2}\sigma^2 t_n\right).$$

3.2.2 The Additional Tracking-Error

We quantify the magnitude of the additional tracking error introduced in the discrete time setting. When considering “additional” tracking error, we refer to the volatility conditioned on given S_n. This is in contrast to the fact that in the continuous time setting, the conditional volatility of $R_{t_n}^A$ is zero conditioned on fixed S_n. Provided that $S_n = S_0\exp\left(n\overline{\mu}(\hat{r}_i)\right)$ and is independent to $s^2(\hat{r}_i)$,

$$\mathbb{E}\left[R_{t_n}^A \middle| S_n\right] \tag{18}$$

$$= \mathbb{E}\left[R_{t_n}^A \;\middle|\, \overline{\mu}(\hat{r}_i)\right]$$

$$\sim \mathbb{E}\left[\exp\left(xn\overline{\mu}(\hat{r}_i) + \frac{x-x^2}{2}\left((n-1)s^2(\hat{r}_i) + n\overline{\mu}(\hat{r}_i)^2\right)\right) - 1\middle|\overline{\mu}(\hat{r}_i)\right]$$

$$= \exp\left(xn\overline{\mu}(\hat{r}_i) + \frac{x-x^2}{2}n\overline{\mu}(\hat{r}_i)^2\right)$$

$$\cdot\, \mathbb{E}\left[\exp\left(\frac{x-x^2}{2}(n-1)s^2(\hat{r}_i)\right)\middle|\overline{\mu}(\hat{r}_i)\right] - 1$$

$$= \exp\left(xn\overline{\mu}(\hat{r}_i) + \frac{x-x^2}{2}n\overline{\mu}(\hat{r}_i)^2\right)\cdot A - 1$$

The conditional mean is slightly different from that in a continuous time setting $\mathbb{E}\left[\mathrm{R}_{\mathrm{t_n}}^{\mathrm{A}} \mid S_n\right] = \exp\left(xn\overline{\mu}(\hat{r}_i)\right) - 1$. We also calculate the conditional standard deviation as

$$\mathrm{std}\left(R_{t_n}^A\middle|\overline{\mu}(\hat{r}_i)\right) \tag{19}$$

$$= \exp\left(xn\overline{\mu}(\hat{r}_i) + \frac{x-x^2}{2}n\overline{\mu}(\hat{r}_i)^2\right)$$

$$\cdot\, \mathrm{std}\left(\exp\left(\frac{x-x^2}{2}(n-1)s^2(\hat{r}_i)\right)\middle|\overline{\mu}(\hat{r}_i)\right)$$

$$= \exp\left(xn\overline{\mu}(\hat{r}_i) + \frac{x-x^2}{2}n\overline{\mu}(\hat{r}_i)^2\right)\cdot \mathrm{std}\left(\exp\left(\frac{x-x^2}{2}\sigma^2\Delta t\mathcal{X}_{n-1}^2\right)\right)$$

$$= \exp\left(xn\overline{\mu}(\hat{r}_i) + \frac{x-x^2}{2}n\overline{\mu}(\hat{r}_i)^2\right)\cdot\sqrt{B-A^2}$$

The mean tracking error conditioned on S_n (or $\overline{\mu}(\hat{r}_i)$, or R_t^S) is therefore

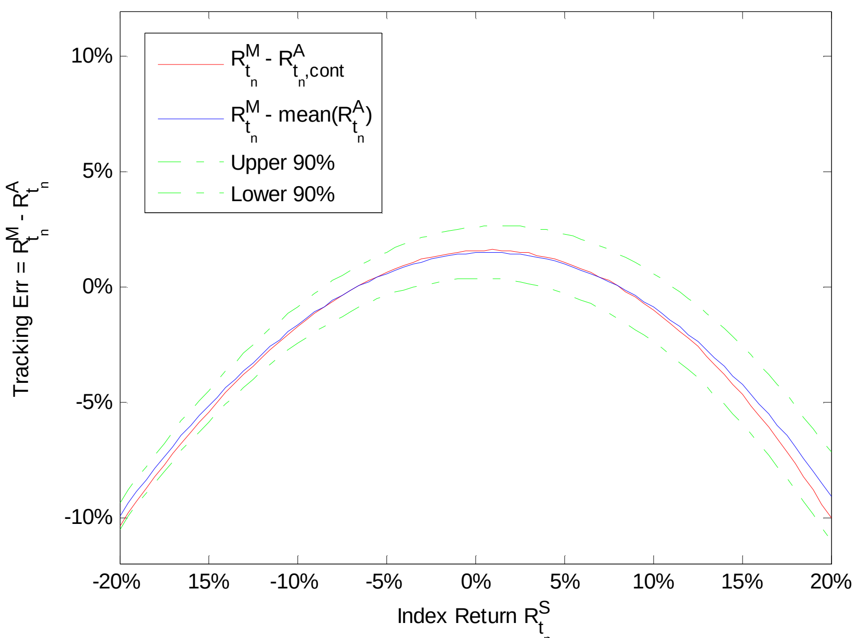
$$\mathbb{E}\left[R_{t_n}^M - R_{t_n}^A \middle| S_n\right] = x\exp\left(xn\overline{\mu}(\hat{r}_i)\right) - \exp\left(xn\overline{\mu}(\hat{r}_i) + \tfrac{x-x^2}{2}n\overline{\mu}(\hat{r}_i)^2\right)\cdot A - x + 1.$$

Since $R_{t_n}^M$ is a constant conditioned on S_n, the conditional standard deviation of tracking error is the same as that of $R_{t_n}^A$.

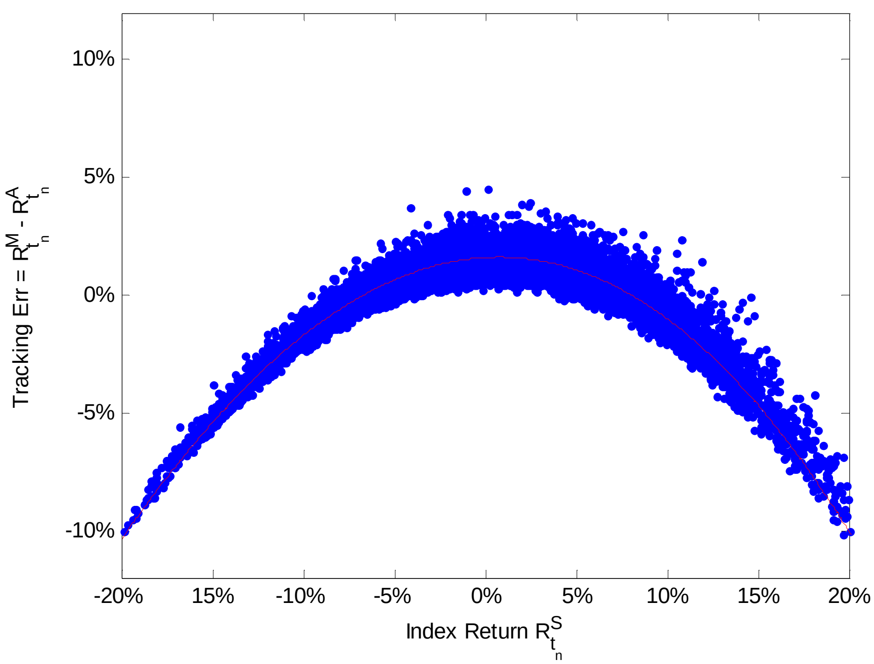
For a concrete example, we set $\mu = 10\%$, $\sigma = 30\%$ and t_n equal to 15 business days (3 weeks, $t_n = 15\Delta t$). We plot in Figure 4 (a) and (b) the tracking error together with the deviation bands. We use $R_{t_n,cont}^A$ to specifically indicate the continuous time result (7): $R_{t_n,cont}^A = \left(\frac{S_t}{S_0}\right)^x \exp\left(\frac{x-x^2}{2}\sigma^2 t\right) - 1$. Figure 4 (b) is based on simulated values.

As we observe in the figure, the conditional mean is not identical to the one in the continuous time results. Even in a 3 week short time period, LETFs yield returns with noise a magnitude of $\pm 3\%$ as a result of daily rebalancing. The magnitude increases sharply with the underlying volatility. For instance, when the underlying volatility changes to 50%, LETFs could yield a $\pm 5\%$ difference in 3 weeks. Considering scenarios when tracking error is small in continuous time setting, i.e., when $R^S_{t_n}$ is roughly $\pm\sigma\sqrt{t_n}$, of the tracking error magnitude, refer to Table 2, which measures the weighted average conditional standard deviation with the distribution of S_n.

Figure 4: Figure (a) plots the tracking error $R^M_{t_n} - R^A_{t_n}$ versus index return $R^S_{t_n}$. The red line is the tracking error in continuous time. The blue line and the green lines are the mean and 90% confidence interval in discrete time. Figure (b) illustrates the same story using simulated results, in which we use 10,000 simulation runs.



(a) Numerical Results



(b) Simulation Results

We analyze the deviation of $R^A_{t_n}$ and $R^A_{t_n,cont}$, the deviation of discrete time versus continuous time. Using Equation (16) and results in Table 1, we calculate the expected value as

$$\begin{aligned}
&\mathbb{E}\left[R^A_{t_n} - R^A_{t_n,cont}\right] \qquad \textbf{(20)}\\
&\sim \exp(x\mu t_n)\left(\exp\left(-\frac{x-x^2}{2}\sigma^2 t_n\right)(1-(x-x^2)\sigma^2 \Delta t)^{-\frac{n-1}{2}} - 1\right)\\
&= \exp(x\mu t_n)\left(\frac{A}{C} - 1\right)
\end{aligned}$$

The standard deviation is calculated as

$$\begin{aligned}
&\mathrm{std}\left(R^A_{t_n} - R^A_{t_n,cont}\right) \qquad \textbf{(21)}\\
&\sim \mathrm{std}\left(\exp\left(xn\bar{\mu}(\hat{r}_i)\right)\left(\exp\left(\frac{x-x^2}{2}(n-1)s^2(\hat{r}_i)\right) - C\right)\right)\\
&= C^{-1}\Big(\exp\left(xt_n(\sigma^2 x + 2\mu)\right)B - 2\exp\left(xt_n(\sigma^2 x + 2\mu)\right)AC\\
&\qquad + \exp\left(xt_n(\sigma^2 x + 2\mu)\right)C^2 - \exp(2x\mu t_n)A^2\\
&\qquad + 2\exp(2x\mu t_n)AC - \exp(2x\mu t_n)C^2\Big)^{\frac{1}{2}}
\end{aligned}$$

Both of the derived quantities are essentially an expectation of the conditional mean and standard deviation, i.e., $\mathbb{E}\left[R_{t_n}^A - R_{t_n,cont}^A\right] = \mathbb{E}\left[\,\mathbb{E}\left[R_{t_n}^A \,\middle|\, S_n\right]\right] - \mathbb{E}\left[R_{t_n,cont}^A\right]$. Table 2 lists the overall standard deviation between $R_{t_n}^A$ and $R_{t_n,cont}^A$. The values are sensitive to the underlying index volatilities and the leverage size. When leverage = -3 the situations become more pronounced. For a 90% confidence band surrounding the mean, we may consider using two times the standard deviation.

Table 2: Standard deviation of LETFs returns $R_{t_n}^A - R_{t_n,cont}^A$. The volatilities are chosen between 10% and 70%, which is the range of observed volatilities from the 90 ProShares LETFs.

	Leverage				
Volatility	-3	-2	-1	2	3
10%	0.12%	0.06%	0.02%	0.02%	0.06%
20%	0.50%	0.25%	0.08%	0.09%	0.26%
30%	1.14%	0.57%	0.19%	0.19%	0.59%
40%	2.06%	1.01%	0.34%	0.35%	1.07%
50%	3.30%	1.60%	0.53%	0.55%	1.71%
60%	4.89%	2.33%	0.76%	0.80%	2.54%
70%	6.90%	3.22%	1.04%	1.10%	3.58%

Finally, we analyze the distribution of the tracking errors. As we have pointed out previously, METFs only depend on ending levels of the underlying index, independent of the sample path. The expected value of the tracking error:

$$\begin{aligned}&\mathbb{E}\left[R_{t_n}^M - R_{t_n}^A\right]\\ &\sim\ \mathbb{E}\left[1 + \mathrm{x}\left(\exp\left(n\overline{\mu}(\hat{r}_i)\right) - 1\right) - \exp\left(xn\overline{\mu}(\hat{r}_i) + \frac{x - x^2}{2}(n-1)s^2(\hat{r}_i)\right)\right]\\ &=\ x\exp(\mu t_n) - (x-1) - \exp(x\mu t_n)\frac{A}{C}\end{aligned} \tag{22}$$

The standard deviation of the tracking error:

$$\begin{aligned}&\mathrm{std}\left(R_{t_n}^M - R_{t_n}^A\right)\\ &\sim\ \mathrm{std}\left(x\exp\left(n\overline{\mu}(\hat{r}_i)\right) - exp\left(xn\overline{\mu}(\hat{r}_i) + \frac{x - x^2}{2}(n-1)s^2(\hat{r}_i)\right)\right)\end{aligned} \tag{23}$$

$$= \begin{pmatrix} \exp(2x\mu t_n - (x - x^2)\sigma^2 t_n)(\exp(x^2\sigma^2 t_n)B - A^2) \\ +x^2\exp(2\mu t_n)(\exp(\sigma^2 t_n) - 1) - \\ \frac{2xA}{C}\left(\exp\left(\frac{1}{2}t_n(x+1)(x\sigma^2 + 2\mu)\right)C - \exp\big((x+1)\mu t_n\big)\right) \end{pmatrix}^{\frac{1}{2}}$$

Table 3: Standard deviation of total tracking error, still using the volatility range from 10% to 70%.

	Leverage				
Volatility	-3	-2	-1	2	3
10%	0.54%	0.27%	0.09%	0.09%	0.28%
20%	2.09%	1.05%	0.35%	0.36%	1.09%
30%	4.68%	2.35%	0.79%	0.81%	2.45%
40%	8.34%	4.17%	1.39%	1.44%	4.41%
50%	13.10%	6.51%	2.17%	2.27%	6.99%
60%	19.01%	9.40%	3.13%	3.30%	10.26%
70%	26.14%	12.83%	4.26%	4.55%	14.30%

The overall tracking error gap is quite large. Comparing Table 2 and Table 3 entry by entry, we observe that the tracking error due to discrete time rebalancing is roughly 25% of the total tracking error.

4 Empirical Analysis

In this section, we use historical data to verify the patterns we derived in the previous section and analyze the potential deviation between the discrete time LETF's return ($R^A_{t_n}$) and its corresponding continuous time return $\left(R^A_{t_n,cont}\right)$. The discrete time return represents the realized total return of a LETF which performs daily portfolio rebalances. As we have shown previously, the continuous time return $R^A_{t_n,cont}$ is governed by Equation (7), which requires three inputs: realized underlying index total return, leverage size, and expected volatility of the underlying index.

We collect 90 LETFs issued by ProShares between June 20, 2006 (the date of their first LETF offering) and June 30, 2009. Investors hold these funds on average for fairly short periods of time: 39 of the funds have average holding periods of less than 1 month[10] and 79 of the funds have an average holding period of less than six months.[11] The fund with the longest average holding period is the ProShares Ultra Short MSCI Mexico (SMK) with 520 days; the fund with the shortest average holding period is ProShares Ultra Pro S&P 500 fund with less than 1 day.

In Table 4 we present the holding periods (Start Date/End Date) during which there is the largest discrepancy in returns (maximum of $R^A_{t_n,cont} - R^A_{t_n}$) between the continuous and discrete cases. The length of these holding periods is fixed at the corresponding average turnover days of each fund. We do not report such time periods for 39 ProShares funds for which the average turnover is less than 1 month. The funds are sorted in an increasing order by average turnover days.

In practice, LETFs rebalance their portfolio daily and generate returns modeled in the discrete time setting. $R^A_{t_n}$ reflects the actual total return for the holding period. For continuous time returns, we use Equation (7) to compute $R^A_{t_n,cont}$. The expected volatility in the formula is unobservable and thus has to be estimated. We calculate the expected volatility as the trailing one year return volatilities before the start date of each period. As noted in the previous sections, the continuous time return $R^A_{t_n,cont}$ should represent the LETF returns when the portfolio is rebalanced continuously.

[10] The average turnover is the average ratio of daily trading volume divided by daily shares outstanding.

[11] See (Guedj, Li, & McCann, 2010) of an analysis of the distribution of holding periods.

Table 4: Of all the 90 ProShares LETFs we analyze, 39 funds have average turnover less than 23 business days (a month). We calculate the holding period (number of turnover days) in which LETFs incur the largest return discrepancy with the continuous time setting. The discrete time return is the fund's realized holding period return. The continuous time return is based on the realized return of the underlying index, leverage, and expected volatility (Equation (7)). The expected volatility (over a trailing year) and realized volatility (during the time period) are also reported.

Fund	Leverage	Start Date	End Date	$R^A_{t_n,cont}$	$R^A_{t_n}$	Expected Volatility	Realized Volatility	Turnover
DUG	-2	10/8/2008	10/10/2008	50.56%	37.78%	22.79%	35.70%	2
QID	-2	9/25/2008	9/29/2008	27.16%	20.16%	14.56%	107.76%	2
QLD	2	4/12/2007	4/16/2007	5.87%	2.11%	13.93%	28.37%	2
TWM	-2	10/10/2008	10/15/2008	8.17%	-1.00%	17.04%	150.84%	3
GLL	-2	3/13/2009	3/18/2009	7.59%	-5.15%	30.02%	16.53%	3
UYG	2	9/15/2008	9/19/2008	41.38%	24.00%	10.69%	140.18%	4
UWM	2	12/1/2008	12/5/2008	22.17%	18.05%	17.04%	64.38%	4
SDS	-2	10/8/2008	10/14/2008	-2.65%	-9.77%	10.40%	127.16%	4
DXD	-2	10/3/2008	10/10/2008	49.18%	39.05%	10.13%	37.86%	5
SCO	-2	11/28/2008	12/5/2008	75.58%	46.52%	46.81%	53.35%	5
UCO	2	12/24/2008	1/2/2009	71.75%	46.49%	46.81%	82.84%	5
SSO	2	9/29/2008	10/6/2008	-8.77%	-13.16%	10.18%	60.89%	5
DDM	2	10/10/2008	10/20/2008	20.17%	11.74%	10.03%	103.70%	6
SRS	-2	11/20/2008	12/1/2008	-11.76%	-36.40%	14.28%	199.84%	6
AGQ	2	5/4/2009	5/13/2009	34.49%	15.10%	50.39%	47.89%	7
MZZ	-2	10/3/2008	10/15/2008	52.12%	38.05%	13.44%	97.37%	8
XPP	2	7/15/2009	7/28/2009	27.56%	20.06%	60.94%	24.84%	9
EET	2	8/19/2009	9/1/2009	4.04%	-2.77%	43.04%	18.88%	9
MVV	2	11/20/2008	12/8/2008	53.56%	48.70%	12.99%	82.78%	11
SKF	-2	9/19/2008	10/7/2008	93.86%	40.51%	10.69%	108.34%	12
DOG	-1	9/23/2008	10/10/2008	28.37%	23.36%	10.03%	54.65%	13
UGL	2	1/29/2009	2/24/2009	20.95%	10.93%	30.02%	25.80%	17
TBT	-2	10/13/2008	11/5/2008	1.03%	-4.45%	12.39%	19.24%	17
USD	2	11/20/2008	12/16/2008	65.09%	50.22%	22.44%	72.61%	17
DIG	2	10/10/2008	11/4/2008	52.90%	36.11%	22.79%	135.72%	17
SDD	-2	10/1/2008	10/27/2008	121.24%	90.67%	15.57%	75.02%	18
EFU	-2	10/2/2008	10/28/2008	88.85%	19.58%	13.21%	78.45%	18
SSG	-2	10/31/2008	12/1/2008	73.37%	48.16%	22.44%	82.13%	20
SKK	-2	9/26/2008	10/27/2008	150.56%	114.49%	17.91%	78.77%	21
SJH	-2	9/26/2008	10/27/2008	140.56%	97.18%	15.97%	79.61%	21
SH	-1	9/26/2008	10/28/2008	28.86%	21.16%	10.18%	88.67%	22

URE	2	11/20/2008	12/23/2008	107.25%	60.98%	14.28%	153.36%	22
SMN	-2	9/25/2008	10/27/2008	233.22%	144.79%	18.46%	106.78%	22
PSQ	-1	9/19/2008	10/22/2008	40.81%	30.89%	13.93%	80.63%	23

There are several possible explanations for the discrepancies noted in Table 4. First, is the difference between discrete and continuous rebalancing in short horizons. This is the argument we develop in the model in the previous section. We demonstrated that $R_{t_n}^A$ is crucially dependent on the realized volatility during the holding period. This volatility could be very different from the expected long run volatility. As the table shows, in most situations, the realized volatility in the holding period becomes large, thus deteriorating the performance of LETFs.

Second, LETFs imperfectly track their reference indexes. The results in Table 4 show that occasionally funds incur large tracking error compared to the underlying index; for example, FXP's return on Oct 15, 2008 was -17.1%, while implied from the index, the continuous rebalanced return should be 28.9%. For those funds with short holding periods (< 3 days) in the table, the funds with large discrepancy in R^A versus $R_{t_n}^A$ is primarily due to the imperfect tracking. In general, LETFs can maintain a stable leverage ratio fluctuating around the target leverage ratio.

Third, index volatility is time-varying and we have assumed constant volatility in the model. During a short time interval, the realized volatility, which crucially affects the performance of LETF returns, would be well-biased from the average volatility. Incorporation of more sophisticated stochastic volatility models to handle this issue could be an informative direction for future research.

These empirical results highlight the two main results of our model. First, continuous and discrete time models provide significantly different assessments of LETFs for short holding periods. LETFs rebalance their position only once a day, and the continuous model implicitly assumes there is a continuous rebalancing. On days where there are large returns there is a non-trivial difference between these assumptions. Second, this example highlights the potentially large difference between theoretical and realized volatility, and its impact on the expectation versus actual deviation between the LETF and a fixed-leveraged investment.

5 Conclusion

Buy and hold investors in LETFs wish to obtain leveraged holding period returns of the underlying index, which we refer to as a fixed-initial-leverage investment strategy. However, LETFs rebalance their portfolios daily, thus obtaining only discretized leveraged exposure. This creates a potentially significant discrepancy in expected and realized returns, even suffering losses in LETFs while the underlying index gains.

We have presented quantitative models of tracking errors between LETFs and similar fixed-initial-leverage investment strategies. In supplement to well analyzed long term tracking deviations, we focus on addressing short term tracking uncertainties. We separated the analysis in continuous time and discrete time settings. Compared to the continuous time settings, discrete time models more accurately portray real life daily rebalancing. Although the two have slight differences, especially for short holding periods, there are additional tracking errors in the discrete time rebalancing. The additional error introduces a 0.2% to 5% difference in holding period returns over 3 weeks. We also qualitatively analyzed the additional error or uncertainties in discrete time models showing that they account for 25% of the total tracking error.

Issuers recommend LETFs for short holding periods. Indeed, almost half of the LETFs surveyed in the paper have average holding periods of less than a month. LETFs are assumed to be able to track a margin account in continuous time models, but in reality, the discrete nature of rebalancing introduces tracking errors which are potentially substantial even during short holding periods. This suggests that the optimal holding time for LETFs may be longer than current recommendations would indicate.

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