Malliavin Calculus in Calculating Delta for Structured Products

Geng Deng Ilan Guedj Sherry Liu Joshua Mallett*

March 21, 2011

Malliavin Calculus [3, 6], also known as *Stochastic Calculus of Variations*, is a useful tool for calculating sensitivities of financial derivatives to a change in its underlying parameters, such as Delta, Vega, and Gamma. In this article, we discuss how to use Malliavin Calculus to calculate Delta for structured products.

Introduction

Delta (Δ) measures the sensitivity of a structured product's market value V to the underlying security's initial price S_0. In risk management, Delta is a useful hedging parameter for controlling the risk of a portfolio.

If we consider the structured product's value as a function of the underlying security's initial price, Delta can be calculated using a combination of simulations and *finite difference* approximations [2]. $\Delta \sim \frac{V(S_0+h)-V(S_0-h)}{2h}$, where h is an infinitesimal change in S_0. However, this approach has two drawbacks. First, there can be a relatively large estimation error in a simulated V. Second, the finite difference method has a difficulty handling non-smooth functions, and most structured products have a non-smooth price function.

Malliavin calculus circumvents these two difficulties by computing the Delta from a different perspective. Let the value of the structured product be $V = \mathbf{E}[e^{-(r+\bar{C})T} f]$, where f is the product's final payoff function, r is the risk-free rate and $\bar{C}$ is the issuer's *credit default swap* spread. Instead of valuing Delta by taking the derivative as shown in Equation (1),

$$\Delta = \frac{\partial V}{\partial S_0} = \mathbf{E}\left[e^{-(r+\bar{C})T} \frac{\partial f}{\partial S_0}\right], \tag{1}$$

Malliavin calculus calculates the Delta as

$$\Delta = \mathbf{E}\left[e^{-(r+\bar{C})T} f \cdot weight\right], \tag{2}$$

where *weight* is a random variable called a 'Malliavin weight'. Using the Malliavin formulation, the Delta can be calculated by simulations without having to subsequently perturb S_0 or use the finite difference method.

In this article, we show how to use Malliavin Calculus to calculate Deltas for three structured product categories: 1) products whose payoffs depend only on the underlying security's ending price $f(S_T)$, 2) products whose payoffs depend on the underlying security's average price $\bar{S}$, $f(\bar{S}) = f\left(\frac{1}{T-T_0}\int_{T_0}^{T} S_t dt\right)$, and 3) products whose payoffs depend on the underlying security's maximum and minimum prices $f\left(\max_0^T S_t, \min_0^T S_t, S_T\right)$.

Calculating Delta

We use a risk-neutral framework and assume the stock price follows a geometric Brownian motion: $dS_t = rS_t dt + \sigma S_t dW_t$. Therefore, $S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}$. Before we describe how to calculate Delta for the structured products we need to define two functional operators used in Malliavin Calculus.

The Operators in Malliavin Calculus

The first operator, D, is the Malliavin derivative applied to functions of stochastic processes. The second operator, δ, is an adjoint operator to D and is called the Skorokhod integral. Like D, δ applies to functions of stochastic processes. δ is essentially the Itô Integral in the form $\delta(u) = \int_0^T u_t dW_t$

*© 2010, Securities Litigation and Consulting Group, Inc. 3998 Fair Ridge Drive, Suite 250 Fairfax, VA 22033. Dr. Geng Deng can be reached at 703.890.0741 or gengdeng@slcg.com, Dr. Ilan Guedj can be reached at 703-865-4020 or ilanguedj@gmail.com, and Mr. Joshua Mallett can be reached at 703.246.9385 or joshuamallet@slcg.com.

where W_t is a standard Brownian motion. The two operators satisfy the 'integration by parts' formulae, also known as the duality principal

$$\mathbf{E}\left[\int_0^T (D_t F) u_t dt\right] = \mathbf{E}[F\delta(u)]. \tag{3}$$

Furthermore, if we let X and Y be two random variables, then

$$\mathbf{E}[f'(X)Y] = \mathbf{E}\left[f(X)\delta\left(\frac{Yh}{\int_0^T h(s)D_s X ds}\right)\right] \tag{4}$$

where $h(t)$ can be any random process. By comparing Equations (2) and (4), we see that the Malliavin weight is

$$weight = \delta\left(\frac{Yh}{\int_0^T h(s)D_s X ds}\right).$$

1. Products whose Payoffs Depend on the Ending Price

We first calculate Deltas for those structured products whose payoffs depend on the linked security's ending price. This is a popular category containing a broad range of structured products such as PLUS, Buffered PLUS and principal protected notes. The payoff function is of the form $f(S_T)$.

By substituting that functional form into Equation (1), we get

$$\begin{aligned}\Delta &= \mathbf{E}\left[e^{-(r+\bar{C})T}\frac{\partial f(S_T)}{\partial S_0}\right] \\ &= \mathbf{E}\left[e^{-(r+\bar{C})T} f'(S_T)\frac{\partial S_T}{\partial S_0}\right].\end{aligned}$$

We then use Equation (4), letting $X = S_T$, $Y = \frac{\partial S_T}{\partial S_0} = \frac{S_T}{S_0}$, and $h(s) = 1/T$ (a constant process). This allows us to simplify as follows:[1]

$$\begin{aligned}\Delta &= e^{-(r+\bar{C})T}\mathbf{E}\left[f(S_T)\delta\left(\frac{S_T}{S_0\int_0^T D_s S_T ds}\right)\right] \\ &= e^{-(r+\bar{C})T}\mathbf{E}\left[f(S_T)\delta\left(\frac{S_T}{S_0\sigma T S_T}\right)\right] \\ &= e^{-(r+\bar{C})T}\mathbf{E}\left[f(S_T)\int_0^T \frac{1}{S_0\sigma T}dW_t\right] \\ &= e^{-(r+\bar{C})T}\mathbf{E}\left[f(S_T)\frac{W_T}{S_0\sigma T}\right].\end{aligned}$$

[1] See for example [3, 5] for a more detailed explanation of the simplification.

Note that the Malliavin weight for this case is:

$$weight = \frac{W_T}{S_0\sigma T}.$$

If the payoff function is simple enough, a closed form-solution for Δ can actually be calculated by evaluating the expectation.

2. Products whose Payoffs Depend on the Average Price

The structured products in this category, including MITTS, have payoffs linked to the average price of the linked security $\bar{S}_T = \frac{1}{T-T_0}\int_{T_0}^T S_t dt$. Applying Malliavin calculus to these structured products is close to calculating Deltas on Asian options.[2] Indeed, it has been proven that the Malliavin approach is superior to the finite difference approach in terms of the convergence rate for Asian options[1].

We again substitute the payoff's functional form into Equation (1) and get

$$\begin{aligned}\Delta &= \mathbf{E}\left[e^{-(r+\bar{C})T}\frac{\partial f(\bar{S}_T)}{\partial S_0}\right] \\ &= \mathbf{E}\left[e^{-(r+\bar{C})T} f'(\bar{S}_T)\frac{\partial \bar{S}_T}{\partial S_0}\right].\end{aligned}$$

Using Equation (4), this time letting $X = \bar{S}_T$, $Y = \frac{\partial \bar{S}_T}{\partial S_0} = \frac{1}{T-T_0}\int_{T_0}^T \frac{\partial}{\partial S_0} S_t dt = \frac{1}{T-T_0}\int_{T_0}^T \frac{S_t}{S_0} dt = \frac{\bar{S}_T}{S_0}$, and $h(t) = S_t$. After some algebraic simplifications we obtain:

$$\begin{aligned}\Delta &= e^{-(r+\bar{C})T}\mathbf{E}\left[f(\bar{S}_T)\delta\left(\frac{\bar{S}_T S_t}{S_0\int_0^T S_s D_s \bar{S}_T ds}\right)\right] \\ &= e^{-(r+\bar{C})T}\mathbf{E}\left[f(\bar{S}_T)\frac{2}{S_0\sigma^2}\left(\frac{S_T - S_0}{T\bar{S}_T} - r + \frac{\sigma^2}{2}\right)\right].\end{aligned}$$

The Malliavin weight in this case is:

$$weight = \frac{2}{S_0\sigma^2}\left(\frac{S_T - S_0}{T\bar{S}_T} - r + \sigma^2/2\right).$$

3. Products whose Payoffs Depend on the Maximum or Minimum Price

This category is also popular, and includes reverse convertibles, ELKS, and absolute return barrier notes. Typically, the payoff function

[2] Asian options are options whose payoff is determined by the average price of the underlying security over a preset period of time.

depends on the linked security's ending price as well as its maximum or minimum price, giving the payoff function the functional form $f(\max S_t, \min S_t, S_T)$. The payoff is similar to that of barrier options or look-back options.

Deriving the Malliavin weight for this category's Delta is more complex than for the first two categories. We take the result directly from [4]:

$$\begin{aligned}\Delta &= e^{-(r+\bar{C})T}\mathbf{E}\Bigg[f(\max X_t, \min X_t, X_T) \\ &\quad \cdot \delta\left(\frac{Z_T}{\int_0^T \Psi(Y_t)dt}\Psi(Y) + \frac{\partial Z_T}{\partial S_0}\right)\Bigg].\end{aligned}$$

A logarithmic transformation needs to be performed on S_t to obtain X_t. Y_t is a dominating process for X_t, for example, $Y_t = \max_{s\leq t}(X_s - X_0) - \min_{s\leq t}(X_s - X_0)$. $\Psi(\cdot)$ is a support function and Z_t is process representing a change of measure along time.

Conclusion

In this article we show how to use Malliavin Calculus to calculate Deltas for a variety of structured product categories. Malliavin Calculus is superior to using a combination of simulations and *finite difference* as a method for calculating Delta. The methods described in this article can be helpful to calculate the Delta of a variety of popular structured products such as PPN, PLUS, Buffered PLUS, MITTS, reverse convertibles, ELKS, absolute return barrier notes, and many others.

References

[1] E. Benhamou, *An application of Malliavin calculus to continuous time Asian options Greeks*, (2000), LSE Working Paper.

[2] M. Broadie and P. Glasserman, *Estimating security price derivatives using simulation*, Management Science **42** (1996), 269–285.

[3] E. Fournié, J. M. Lasry, j. Lebuchoux, P. L. Lions, and N. Touzi, *Applications of Malliavin calculus to Monte Carlo methods in finance*, Finance Stochast. **3** (1999), 391–412.

[4] E. Gobet and A. Kohatsu-Higa, *Computations of Greeks for barrier and look-back options using Malliavin calculus*, Elect. Comm. in Probab. **8** (2003), 51–62.

[5] M. Montero and A. Kohatsu-Higa, *Malliavin calculus applied to finance*, (2002), Working paper.

[6] D. Nualart, *Malliavin calculus and related topics. (probability and its applications)*, Springer, Berlin Heidelber New York, 1995.